Lecture 5 – Limits at Infinity and Asymptotes

- **Limits at Infinity**
  - Horizontal Asymptotes

- **Limits at Infinity for Polynomials**
  - Limit of a Reciprocal Power
  - The End Behavior of a Polynomial
  - Evaluating the Limit at Infinity of a Polynomial
  - Example 17 – Limit of Rational Function
  - Example 18 – More Rational Functions

- **Limits at Infinity for the Exponential Function**
  - Example 19 – Limits of Exponential Functions

- **Function Dominance**
  - Example 20 – Exponential and Power Functions

- **More on Asymptotes**
  - Vertical Asymptotes and Infinite Limits
  - Horizontal Asymptotes and Limits at Infinity
  - Example 21 – Investigating Asymptotes
Limits at Infinity

If a function $f$ has a domain that is unbounded, that is, one of the endpoints of its domain is $\pm \infty$, we can determine the long term behavior of the function using a limit at infinity.
Limits at Infinity

If a function $f$ has a domain that is unbounded, that is, one of the endpoints of its domain is $\pm\infty$, we can determine the long term behavior of the function using a limit at infinity.

Definition (Limits at Infinity)
Limits at Infinity

If a function $f$ has a domain that is unbounded, that is, one of the endpoints of its domain is $\pm\infty$, we can determine the long term behavior of the function using a limit at infinity.

**Definition (Limits at Infinity)**

We say that

$$\lim_{{x \to \infty}} f(x) = L$$

if $f(x)$ can be made arbitrarily close to $L$ by making $x$ sufficiently large and positive.
Limits at Infinity

If a function $f$ has a domain that is unbounded, that is, one of the endpoints of its domain is $\pm \infty$, we can determine the long term behavior of the function using a limit at infinity.

**Definition (Limits at Infinity)**

We say that

$$\lim_{x \to \infty} f(x) = L$$

if $f(x)$ can be made arbitrarily close to $L$ by making $x$ sufficiently large and positive, and we say that

$$\lim_{x \to -\infty} f(x) = M$$

if $f(x)$ can be made arbitrarily close to $M$ by making $x$ sufficiently large and negative.
Horizontal Asymptotes
Horizontal Asymptotes

If

$$\lim_{x \to \infty} f(x) = L$$

then the graph of $f$ has a **horizontal asymptote** along the line $y = L$ on the right.
Horizontal Asymptotes

If

\[ \lim_{x \to \infty} f(x) = L \]

then the graph of \( f \) has a **horizontal asymptote** along the line \( y = L \) **on the right**, and if

\[ \lim_{x \to -\infty} f(x) = M \]

then the graph of \( f \) has a horizontal asymptote along the line \( y = M \) **on the left**.
Horizontal Asymptotes

If
\[ \lim_{x \to \infty} f(x) = L \]
then the graph of \( f \) has a **horizontal asymptote** along the line \( y = L \) **on the right**, and if
\[ \lim_{x \to -\infty} f(x) = M \]
then the graph of \( f \) has a horizontal asymptote along the line \( y = M \) **on the left**.

If \( f \) has an unbounded domain, but \( f(x) \) does not approach a finite value as \( x \) goes to \( \pm \infty \), there are two possibilities:
Horizontal Asymptotes

If

$$\lim_{x \to \infty} f(x) = L$$

then the graph of $f$ has a \textbf{horizontal asymptote} along the line $y = L$ \textbf{on the right}, and if

$$\lim_{x \to -\infty} f(x) = M$$

then the graph of $f$ has a horizontal asymptote along the line $y = M$ \textbf{on the left}.

If $f$ has an unbounded domain, but $f(x)$ does not approach a finite value as $x$ goes to $\pm\infty$, there are two possibilities:

- the limit at infinity is infinite
Horizontal Asymptotes

If
\[ \lim_{x \to \infty} f(x) = L \]
then the graph of \( f \) has a **horizontal asymptote** along the line \( y = L \) **on the right**, and if
\[ \lim_{x \to -\infty} f(x) = M \]
then the graph of \( f \) has a horizontal asymptote along the line \( y = M \) **on the left**.

If \( f \) has an unbounded domain, but \( f(x) \) does not approach a finite value as \( x \) goes to \( \pm \infty \), there are two possibilities:

- the limit at infinity is infinite
- the limit at infinity does not exist
If

$$\lim_{x \to \infty} f(x) = L$$

then the graph of $f$ has a **horizontal asymptote** along the line $y = L$ **on the right**,

and if

$$\lim_{x \to -\infty} f(x) = M$$

then the graph of $f$ has a horizontal asymptote along the line $y = M$ **on the left**.

If $f$ has an unbounded domain, but $f(x)$ does not approach a finite value as $x$ goes to $\pm \infty$, there are two possibilities:

- the limit at infinity is infinite
- the limit at infinity does not exist

In either case, the graph of $f$ does not have a horizontal asymptote.
Limit of a Reciprocal Power
Limit of a Reciprocal Power

If \( r > 0 \), then

\[
\lim_{x \to \infty} \frac{1}{x^r} = 0
\]
### Limit of a Reciprocal Power

If \( r > 0 \), then

\[
\lim_{x \to \infty} \frac{1}{x^r} = 0
\]

and

\[
\lim_{x \to -\infty} \frac{1}{x^r} = 0
\]
Limit of a Reciprocal Power

If \( r > 0 \), then

\[
\lim_{x \to \infty} \frac{1}{x^r} = 0
\]

and

\[
\lim_{x \to -\infty} \frac{1}{x^r} = 0
\]

if \( r \) is a positive integer or is a fraction that results is an even root.
The End Behavior of a Polynomial

Limits at Infinity for Polynomials

Limits at Infinity for the Exponential Function

Function Dominance

More on Asymptotes
The End Behavior of a Polynomial

Limits at Infinity for Polynomials

For the \( n^{th} \) degree polynomial function

\[
p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0
\]

the end behavior is given by the limit at infinity
The End Behavior of a Polynomial

Limits at Infinity for Polynomials

For the $n^{th}$ degree polynomial function

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

the end behavior is given by the limit at infinity

$$\lim_{x \to \pm \infty} p(x)$$
The End Behavior of a Polynomial

Limits at Infinity for Polynomials

For the $n^{th}$ degree polynomial function

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

the end behavior is given by the limit at infinity

$$\lim_{x \to \pm \infty} p(x)$$

which is always infinite and whose sign depends only on the sign of $a_n$ and on whether $n$ even or odd.
Evaluating the Limit at Infinity of a Polynomial

The limit at infinity of a polynomial is

\[
\lim_{x \to \pm \infty} p(x) = \lim_{x \to \pm \infty} \left( a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \right)
\]
Evaluating the Limit at Infinity of a Polynomial

The limit at infinity of a polynomial is

$$\lim_{x \to \pm \infty} p(x) = \lim_{x \to \pm \infty} \left( a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \right)$$

$$= \lim_{x \to \pm \infty} x^n \left( a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right)$$
Evaluating the Limit at Infinity of a Polynomial

The at infinity of a polynomial is

$$\lim_{x \to \pm \infty} p(x) = \lim_{x \to \pm \infty} \left( a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \right)$$

$$= \lim_{x \to \pm \infty} x^n \left( a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right)$$

Using the the limit at infinity of a reciprocal power, gives
Evaluating the Limit at Infinity of a Polynomial

The limit at infinity of a polynomial is

$$\lim_{x \to \pm \infty} p(x) = \lim_{x \to \pm \infty} \left( a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \right)$$

$$= \lim_{x \to \pm \infty} x^n \left( a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right)$$

Using the limit at infinity of a reciprocal power, gives

$$\lim_{x \to \pm \infty} p(x) = \lim_{x \to \pm \infty} x^n \left( a_n + \frac{a_{n-1}}{x} \to 0 + \cdots + \frac{a_2}{x^{n-2}} \to 0 + \frac{a_1}{x^{n-1}} \to 0 + \frac{a_0}{x^n} \to 0 \right)$$
Evaluating the Limit at Infinity of a Polynomial

The limit at infinity of a polynomial is

\[
\lim_{x \to \pm \infty} p(x) = \lim_{x \to \pm \infty} \left( a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \right)
\]

\[
= \lim_{x \to \pm \infty} x^n \left( a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right)
\]

Using the limit at infinity of a reciprocal power, gives

\[
\lim_{x \to \pm \infty} p(x) = \lim_{x \to \pm \infty} x^n \left( a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right)
\]

All terms except the first go to zero.
Evaluating the Limit at Infinity of a Polynomial

The at infinity of a polynomial is

$$\lim_{x \to \pm\infty} p(x) = \lim_{x \to \pm\infty} \left( a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \right)$$

$$= \lim_{x \to \pm\infty} x^n \left( a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right)$$

Using the the limit at infinity of a reciprocal power, gives

$$\lim_{x \to \pm\infty} p(x) = \lim_{x \to \pm\infty} x^n \left( a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right) \rightarrow 0$$

All terms except the first go to zero. Thus,

$$\lim_{x \to \pm\infty} p(x) = a_n \lim_{x \to \pm\infty} x^n$$
Evaluating the Limit at Infinity of a Polynomial

Now, if $n > 0$, then

$$n \text{ even } \Rightarrow \lim_{x \to \pm \infty} x^n = \infty \quad \text{and} \quad n \text{ odd } \Rightarrow \lim_{x \to \pm \infty} x^n = \pm \infty$$
Evaluating the Limit at Infinity of a Polynomial

Now, if \( n > 0 \), then

\[
\text{n even } \Rightarrow \lim_{x \to \pm \infty} x^n = \infty \quad \text{and} \quad \text{n odd } \Rightarrow \lim_{x \to \pm \infty} x^n = \pm \infty
\]

Thus,

\[
\lim_{x \to \infty} a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = \begin{cases} 
+\infty & \text{if } a_n > 0 \\
-\infty & \text{if } a_n < 0
\end{cases}
\]
Evaluating the Limit at Infinity of a Polynomial

Now, if $n > 0$, then

\[
\text{n even } \Rightarrow \lim_{x \to \pm \infty} x^n = \infty \quad \text{and} \quad \text{n odd } \Rightarrow \lim_{x \to \pm \infty} x^n = \pm \infty
\]

Thus,

\[
\lim_{x \to \infty} a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = \begin{cases} 
+\infty & \text{if } a_n > 0 \\
-\infty & \text{if } a_n < 0 
\end{cases}
\]

and

\[
\lim_{x \to -\infty} a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 =
\begin{cases} 
+\infty & \text{if } n \text{ even } \& a_n > 0 \text{ or } n \text{ odd } \& a_n < 0 \\
-\infty & \text{if } n \text{ even } \& a_n < 0 \text{ or } n \text{ odd } \& a_n > 0 
\end{cases}
\]
Example 17 – Limit of Rational Function

Evaluate the limit

\[
\lim_{x \to \infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3}
\]
**Solution: Example 17**

To make use of the limits just discussed, the expression must involve terms with reciprocal powers. Thus divide the numerator and denominator by the highest power of \( x \) in the denominator, here \( x^3 \). This gives

\[
\lim_{x \to \infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} = \lim_{x \to \infty} \frac{4 - \frac{3}{x} + \frac{2}{x^2} - \frac{4}{x^3}}{5 + \frac{2}{x} - \frac{4}{x^2} + \frac{3}{x^3}}
\]
Solution: Example 17

To make use of the limits just discussed, the expression must involve terms with reciprocal powers. Thus divide the numerator and denominator by the highest power of $x$ in the denominator, here $x^3$. This gives

$$
\lim_{x \to \infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} = \lim_{x \to \infty} \frac{\frac{4}{x} - \frac{3}{x^2} + \frac{2}{x^3} - \frac{4}{x^4}}{\frac{5}{x^3} + \frac{2}{x^2} - \frac{4}{x^3} + \frac{3}{x^4}}
$$

$$
= \lim_{x \to \infty} \frac{4 \to 0 - 3 \to 0 + 2 \to 0 - 4 \to 0}{5 \to 0 + 2 \to 0 - 4 \to 0 + 3 \to 0}
$$

Clint Lee

Math 112 Lecture 5: Limits at Infinity and Asymptotes
Example 17 – Limit of Rational Function

Solution: Example 17

To make use of the limits just discussed, the expression must involve terms with reciprocal powers. Thus divide the numerator and denominator by the highest power of $x$ in the denominator, here $x^3$. This gives

$$
\lim_{x \to \infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} = \lim_{x \to \infty} \frac{4 - \frac{3}{x} + \frac{2}{x^2} - \frac{4}{x^3}}{5 + \frac{2}{x} - \frac{4}{x^2} + \frac{3}{x^3}}
$$

$$
= \lim_{x \to \infty} \frac{4 - \frac{3}{x} + \frac{2}{x^2} - \frac{4}{x^3}}{5 + \frac{2}{x} - \frac{4}{x^2} + \frac{3}{x^3}}
$$

$$
= \frac{4}{5}
$$
Solution: Example 17 continue

An identical argument for \( x \to -\infty \) shows that

\[
\lim_{x \to -\infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} = \frac{4}{5}
\]

Hence the graph of the rational function

\[
r(x) = \frac{4x^3 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3}
\]

has a horizontal asymptote along the
Solution: Example 17 continue

An identical argument for \( x \to -\infty \) shows that

\[
\lim_{x \to -\infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} = \frac{4}{5}
\]

Hence the graph of the rational function

\[
r(x) = \frac{4x^3 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3}
\]

has a horizontal asymptote along the line \( y = \frac{4}{5} \) on both the right and the left.
Solution: Example 17 continue

An identical argument for \( x \to -\infty \) shows that

\[
\lim_{x \to -\infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} = \frac{4}{5}
\]

Hence the graph of the rational function

\[
r(x) = \frac{4x^3 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3}
\]

has a horizontal asymptote along the line \( y = \frac{4}{5} \) on both the right and the left.
Solution: Example 17 continue

An identical argument for $x \to -\infty$ shows that

$$\lim_{x \to -\infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} = \frac{4}{5}$$

Hence the graph of the rational function

$$r(x) = \frac{4x^3 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3}$$

has a horizontal asymptote along the line $y = \frac{4}{5}$ on both the right and the left.

Horizontal Asymptotes of a Rational Function

In general, for a rational function in which the degrees of polynomials in the numerator and denominator are equal, the limit at $\pm \infty$ equals the ratio of the coefficients of the highest powers, call it $R$, 

$$R$$

Solution: Example 17 continue

An identical argument for $x \to -\infty$ shows that

$$\lim_{x \to -\infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} = \frac{4}{5}$$

Hence the graph of the rational function

$$r(x) = \frac{4x^3 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3}$$

has a horizontal asymptote along the line $y = \frac{4}{5}$ on both the right and the left.

**Horizontal Asymptotes of a Rational Function**

In general, for a rational function in which the degrees of polynomials in the numerator and denominator are equal, the limit at $\pm\infty$ equals the ratio of the coefficients of the highest powers, call it $R$, and the function has a horizontal asymptote along a line $y = R$ on both sides.
Example 18 – More Rational Functions

Evaluate the limits

(a) \( \lim_{x \to \pm \infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^4 + 2x^2 - 4x + 3} \)

(b) \( \lim_{x \to \pm \infty} \frac{4x^4 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} \)
Example 18 – More Rational Functions

Evaluate the limits

(a) \( \lim_{x \to \pm \infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^4 + 2x^2 - 4x + 3} \)  
(b) \( \lim_{x \to \pm \infty} \frac{4x^4 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} \)

Solution:

(a) \( \lim_{x \to \pm \infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^4 + 2x^2 - 4x + 3} \)
Example 18 – More Rational Functions

Evaluate the limits

(a) \[ \lim_{x \to \pm \infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^4 + 2x^2 - 4x + 3} \]

(b) \[ \lim_{x \to \pm \infty} \frac{4x^4 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} \]

Solution:

(a) \[ \lim_{x \to \pm \infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^4 + 2x^2 - 4x + 3} = \lim_{x \to \pm \infty} \frac{4}{5} - \frac{3}{x^2} + \frac{2}{x^3} - \frac{4}{x^4} \]

(b) \[ \lim_{x \to \pm \infty} \frac{4x^4 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} = \lim_{x \to \pm \infty} \frac{4}{5} + \frac{2}{x^2} - \frac{4}{x^3} + \frac{3}{x^4} \]
Example 18 – More Rational Functions

Evaluate the limits

(a) \[ \lim_{x \to \pm \infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^4 + 2x^2 - 4x + 3} \]

(b) \[ \lim_{x \to \pm \infty} \frac{4x^4 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} \]

Solution:

(a) \[ \lim_{x \to \pm \infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^4 + 2x^2 - 4x + 3} = \lim_{x \to \pm \infty} \frac{4}{5} \cdot \frac{x^3}{x^4} - \frac{3}{5} \cdot \frac{x^2}{x^4} + \frac{2}{5} \cdot \frac{x}{x^4} - \frac{4}{3} \cdot \frac{1}{x^4} = 0 \]
Evaluate the limits

(a) \[ \lim_{{x \to \pm \infty}} \frac{4x^3 - 3x^2 + 2x - 4}{5x^4 + 2x^2 - 4x + 3} \]

(b) \[ \lim_{{x \to \pm \infty}} \frac{4x^4 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} \]

Solution:

(a) \[ \lim_{{x \to \pm \infty}} \frac{4x^3 - 3x^2 + 2x - 4}{5x^4 + 2x^2 - 4x + 3} = \lim_{{x \to \pm \infty}} \frac{4 \frac{x}{x^3} - 3 \frac{x}{x^2} + 2 \frac{1}{x^2} - 4 \frac{1}{x^3}}{5 + 2 \frac{1}{x^2} - 3 \frac{1}{x^3} + 3 \frac{1}{x^4}} = 0 \]

(b) \[ \lim_{{x \to \pm \infty}} \frac{4x^4 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} \]
Example 18 – More Rational Functions

Evaluate the limits

\[(a) \lim_{x \to \pm \infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^4 + 2x^2 - 4x + 3}\]

\[(b) \lim_{x \to \pm \infty} \frac{4x^4 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3}\]

Solution:

(a) \[\lim_{x \to \pm \infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^4 + 2x^2 - 4x + 3} = \lim_{x \to \pm \infty} \frac{4}{x} - \frac{3}{x^2} + \frac{2}{x^3} - \frac{4}{x^4} = 0\]

(b) \[\lim_{x \to \pm \infty} \frac{4x^4 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} = \lim_{x \to \pm \infty} \frac{4x}{x} - \frac{2}{x^2} - \frac{4}{x^3} + \frac{3}{x} - \frac{4}{x^2} + \frac{3}{x^3} = 0\]
Example 18 – More Rational Functions

Evaluate the limits

(a) \( \lim_{x \to \pm \infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^4 + 2x^2 - 4x + 3} \)

(b) \( \lim_{x \to \pm \infty} \frac{4x^4 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} \)

Solution:

(a) \( \lim_{x \to \pm \infty} \frac{4x^3 - 3x^2 + 2x - 4}{5x^4 + 2x^2 - 4x + 3} = \lim_{x \to \pm \infty} \frac{4}{5} - \frac{3}{x^2} + \frac{2}{x^3} - \frac{4}{x^4} = 0 \)

(b) \( \lim_{x \to \pm \infty} \frac{4x^4 - 3x^2 + 2x - 4}{5x^3 + 2x^2 - 4x + 3} = \lim_{x \to \pm \infty} \frac{4x - \frac{3}{x}}{5} + \frac{2}{x^2} - \frac{4}{x^3} = \pm \infty \)
Limits at Infinity for the Exponential Function

The graph of the natural exponential function $f(x) = e^x$ looks like this.
Limits at Infinity for the Exponential Function

The graph of the **natural exponential function** \( f(x) = e^x \) looks like this.
Limits at Infinity for the Exponential Function

The graph of the **natural exponential** function $f(x) = e^x$ looks like this. The graph of the function $g(x) = e^{-x}$ is obtained by reflecting across the $y$-axis.
Limits at Infinity for the Exponential Function

The graph of the **natural exponential** function \( f(x) = e^x \) looks like this. The graph of the function \( g(x) = e^{-x} \) is obtained by reflecting across the \( y \)-axis. It looks like this.
Limits at Infinity for the Exponential Function
Limits at Infinity for the Exponential Function

From the graphs just shown

\[ \lim_{x \to \infty} e^x = \infty \]
Limits at Infinity for the Exponential Function

From the graphs just shown

\[ \lim_{x \to \infty} e^x = \infty \quad \text{and} \quad \lim_{x \to -\infty} e^x = 0 \]
Limits at Infinity for the Exponential Function

From the graphs just shown

\[ \lim_{x \to \infty} e^x = \infty \quad \text{and} \quad \lim_{x \to -\infty} e^x = 0 \]

and
Limits at Infinity for the Exponential Function

From the graphs just shown

\[ \lim_{x \to \infty} e^x = \infty \quad \text{and} \quad \lim_{x \to -\infty} e^x = 0 \]

and

\[ \lim_{x \to -\infty} e^{-x} \]
From the graphs just shown

\[
\lim_{x \to \infty} e^x = \infty \quad \text{and} \quad \lim_{x \to -\infty} e^x = 0
\]

and

\[
\lim_{x \to -\infty} e^{-x} = \infty
\]
Limits at Infinity for the Exponential Function

From the graphs just shown

\[
\lim_{x \to \infty} e^x = \infty \quad \text{and} \quad \lim_{x \to -\infty} e^x = 0
\]

and

\[
\lim_{x \to -\infty} e^{-x} = \infty \quad \text{and} \quad \lim_{x \to \infty} e^{-x} = 0
\]
Example 19 – Limits of Exponential Functions

Evaluate the limits

\[(a) \quad \lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2} \quad \quad (b) \quad \lim_{x \to \infty} \frac{2e^x + 1}{e^x + 2}\]
Evaluate the limits

(a) \[ \lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2} \]

(b) \[ \lim_{x \to \infty} \frac{2e^x + 1}{e^x + 2} \]

Solution:
(a) Since \[ \lim_{x \to -\infty} e^x = 0 \] we have
Example 19 – Limits of Exponential Functions

Evaluate the limits

(a) \( \lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2} \)

(b) \( \lim_{x \to \infty} \frac{2e^x + 1}{e^x + 2} \)

Solution:

(a) Since \( \lim_{x \to -\infty} e^x = 0 \) we have

\[
\lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2}
\]
Example 19 – Limits of Exponential Functions

Evaluate the limits

(a) \( \lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2} \)

(b) \( \lim_{x \to \infty} \frac{2e^x + 1}{e^x + 2} \)

Solution:

(a) Since \( \lim_{x \to -\infty} e^x = 0 \) we have

\[
\lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2} = \frac{2 \cdot 0 + 1}{0 + 2} = \frac{1}{2}
\]
Example 19 – Limits of Exponential Functions

Evaluate the limits

(a) \( \lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2} \)

(b) \( \lim_{x \to \infty} \frac{2e^x + 1}{e^x + 2} \)

Solution:

(a) Since \( \lim_{x \to -\infty} e^x = 0 \) we have

\[
\lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2} = \frac{2 \cdot 0 + 1}{0 + 2} = \frac{1}{2}
\]
Example 19 – Limits of Exponential Functions

Evaluate the limits

(a) \( \lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2} \)

(b) \( \lim_{x \to \infty} \frac{2e^x + 1}{e^x + 2} \)

Solution:

(a) Since \( \lim_{x \to -\infty} e^x = 0 \) we have

\[
\lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2} = \frac{2 \cdot 0 + 1}{0 + 2} = \frac{1}{2}
\]

(b) Now we use \( \lim_{x \to \infty} e^{-x} = 0 \). To do the we first multiply numerator and denominator by \( e^{-x} \) to give
Example 19 – Limits of Exponential Functions

Evaluate the limits

(a) \( \lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2} \)

(b) \( \lim_{x \to \infty} \frac{2e^x + 1}{e^x + 2} \)

Solution:

(a) Since \( \lim_{x \to -\infty} e^x = 0 \) we have

\[
\lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2} = \frac{2 \cdot 0 + 1}{0 + 2} = \frac{1}{2}
\]

(b) Now we use \( \lim_{x \to \infty} e^{-x} = 0 \). To do this we first multiply numerator and denominator by \( e^{-x} \) to give

\[
\lim_{x \to \infty} \frac{2e^x + 1}{e^x + 2}
\]
Example 19 – Limits of Exponential Functions

Evaluate the limits

(a) \[ \lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2} \]

(b) \[ \lim_{x \to \infty} \frac{2e^x + 1}{e^x + 2} \]

Solution:

(a) Since \( \lim_{x \to -\infty} e^x = 0 \) we have

\[ \lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2} = \frac{2 \cdot 0 + 1}{0 + 2} = \frac{1}{2} \]

(b) Now we use \( \lim_{x \to \infty} e^{-x} = 0 \). To do the we first multiply numerator and denominator by \( e^{-x} \) to give

\[ \lim_{x \to \infty} \frac{2e^x + 1}{e^x + 2} = \lim_{x \to \infty} \left( \frac{2e^x + 1}{e^x + 2} \right) \left( \frac{e^{-x}}{e^{-x}} \right) \]
Example 19 – Limits of Exponential Functions

Evaluate the limits

(a) \[ \lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2} \]

(b) \[ \lim_{x \to \infty} \frac{2e^x + 1}{e^x + 2} \]

Solution:

(a) Since \( \lim_{x \to -\infty} e^x = 0 \) we have

\[ \lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2} = \frac{2 \cdot 0 + 1}{0 + 2} = \frac{1}{2} \]

(b) Now we use \( \lim_{x \to \infty} e^{-x} = 0 \). To do the we first multiply numerator and denominator by \( e^{-x} \) to give

\[ \lim_{x \to \infty} \frac{2e^x + 1}{e^x + 2} = \lim_{x \to \infty} \left( \frac{2e^x + 1}{e^x + 2} \right) \left( \frac{e^{-x}}{e^{-x}} \right) = \lim_{x \to \infty} \frac{2 + e^{-x}}{1 + 2e^{-x}} \]
Example 19 – Limits of Exponential Functions

Evaluate the limits

(a) \( \lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2} \)

(b) \( \lim_{x \to \infty} \frac{2e^x + 1}{e^x + 2} \)

Solution:

(a) Since \( \lim_{x \to -\infty} e^x = 0 \) we have

\[
\lim_{x \to -\infty} \frac{2e^x + 1}{e^x + 2} = \frac{2 \cdot 0 + 1}{0 + 2} = \frac{1}{2}
\]

(b) Now we use \( \lim_{x \to \infty} e^{-x} = 0 \). To do this we first multiply numerator and denominator by \( e^{-x} \) to give

\[
\lim_{x \to \infty} \frac{2e^x + 1}{e^x + 2} = \lim_{x \to \infty} \left( \frac{2e^x + 1}{e^x + 2} \right) \left( \frac{e^{-x}}{e^{-x}} \right) = \lim_{x \to \infty} \frac{2 + e^{-x}}{1 + 2e^{-x}} = 2
\]
Solution: Example 19 continued

The graph of the function $f(x) = \frac{2e^x + 1}{e^x + 2}$ has a horizontal asymptotes along $y = \frac{1}{2}$ on the left and $y = 2$ on the right. Its graph looks like this.
Solution: Example 19 continued

The graph of the function \( f(x) = \frac{2e^x + 1}{e^x + 2} \) has a horizontal asymptotes along \( y = \frac{1}{2} \) on the left and \( y = 2 \) on the right. Its graph looks like this.
Horizontal Asymptotes for Exponential Functions
Horizontal Asymptotes for Functions Involving the Exponential Function

For a function involving an exponential function the limits at $+\infty$ and $-\infty$ can be different.
Horizontal Asymptotes for Functions Involving the Exponential Function

For a function involving an exponential function the limits at $+\infty$ and $-\infty$ can be different. This means that such functions can have a different horizontal asymptote on the right ($x \to \infty$) and on the left ($x \to -\infty$).
Function Dominance

When comparing functions, an important question is: “which function grows faster as $x$ gets large?” This question can be answered using limits at infinity.
Function Dominance

When comparing functions, an important question is: “which function grows faster as $x$ gets large?” This question can be answered using limits at infinity.
When comparing functions, an important question is: “which function grows faster as $x$ gets large?” This question can be answered using limits at infinity.

For two functions $f$ and $g$, if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

we say that $g$ dominates $f$ as $x$ gets large and positive.
Function Dominance

When comparing functions, an important question is: “which function grows faster as \( x \) gets large?” This question can be answered using limits at infinity.

**Function Dominance**

For two functions \( f \) and \( g \), if

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0
\]

we say that \( g \) dominates \( f \) as \( x \) gets large and positive. Equivalently, we have

\[
\lim_{x \to \infty} \frac{g(x)}{f(x)} = \infty
\]
Example 20 – Exponential and Power Functions

Consider the family of functions

\[ f(x) = x^n e^{-x} = \frac{x}{e^x} \quad \text{for } x \geq 0 \]

where the parameter \( n \) is a positive integer.
Example 20 – Exponential and Power Functions

Consider the family of functions

\[ f(x) = x^n e^{-x} = \frac{x}{e^x} \quad \text{for } x \geq 0 \]

where the parameter \( n \) is a positive integer.

The graph of the single member of the family with \( n = 1 \) looks like this.
Example 20 – Exponential and Power Functions

Consider the family of functions

\[ f(x) = x^n e^{-x} = \frac{x}{e^x} \quad \text{for } x \geq 0 \]

where the parameter \( n \) is a positive integer.

The graph of the two members of the family with \( n = 1 \) and \( n = 2 \) look like this.
Consider the family of functions

\[ f(x) = x^n e^{-x} = \frac{x}{e^x} \quad \text{for } x \geq 0 \]

where the parameter \( n \) is a positive integer.

The graph of the three members of the family with \( n = 1, n = 2, \) and \( n = 3 \) look like this.
Consider the family of functions

\[ f(x) = x^n e^{-x} = \frac{x}{e^x} \quad \text{for } x \geq 0 \]

where the parameter \( n \) is a positive integer.

The graph of the four members of the family with \( n = 1, n = 2, n = 3, \) and \( n = 4 \) look like this.
Example 20 – Exponential and Power Functions

Consider the family of functions

\[ f(x) = x^n e^{-x} = \frac{x}{e^x} \quad \text{for } x \geq 0 \]

where the parameter \( n \) is a positive integer.

The graph of the four members of the family with \( n = 1, \ n = 2, \ n = 3, \) and \( n = 4 \) look like this.

Based on these graphs what can you conclude about the dominance relationship between the exponential function \( e^x \) and the power function \( x^n \)?
Example 20 – Exponential and Power Functions

Solution: Example 20

From the graphs it appears that for any integer \( n \geq 1 \) we have

\[
\lim_{x \to \infty} \frac{x^n}{e^x} = 0
\]
Solution: Example 20

From the graphs it appears that for any integer $n \geq 1$ we have

$$\lim_{x \to \infty} \frac{x^n}{e^x} = 0$$

since for each of the graphs the $x$-axis, $y = 0$, is a horizontal asymptote.
Solution: Example 20

From the graphs it appears that for any integer \( n \geq 1 \) we have

\[
\lim_{x \to \infty} \frac{x^n}{e^x} = 0
\]

since for each of the graphs the \( x \)-axis, \( y = 0 \), is a horizontal asymptote.

Thus, we conclude that the exponential function \( e^x \) dominates \( x^n \) as \( x \to \infty \).
Solution: Example 20

From the graphs it appears that for any integer $n \geq 1$ we have

$$\lim_{x \to \infty} \frac{x^n}{e^x} = 0$$

since for each of the graphs the $x$-axis, $y = 0$, is a horizontal asymptote.

Thus, we conclude that the exponential function $e^x$ dominates $x^n$ as $x \to \infty$.

In other words, the exponential function grows more rapidly than the power function as $x$ gets large and positive.
Solution: Example 20

From the graphs it appears that for any integer $n \geq 1$ we have

$$\lim_{x \to \infty} \frac{x^n}{e^x} = 0$$

since for each of the graphs the $x$-axis, $y = 0$, is a horizontal asymptote.

Thus, we conclude that the exponential function $e^x$ dominates $x^n$ as $x \to \infty$.

In other words, the exponential function grows more rapidly than the power function as $x$ gets large and positive.

Further, from the graphs it appears that as $n$ gets larger, we must go to larger $x$ values for the dominance to take effect.
Vertical Asymptotes and Infinite Limits
Vertical Asymptotes and Infinite Limits

Where Are Vertical Asymptotes?
Where Are Vertical Asymptotes?

Where the function has an infinite limit.
Vertical Asymptotes and Infinite Limits

Where Are Vertical Asymptotes?

Where the function has an infinite limit.

- There may be no vertical asymptotes, or many.
Vertical Asymptotes and Infinite Limits

Where Are Vertical Asymptotes?

Where the function has an **infinite limit**.

- There may be no vertical asymptotes, or many.
- A rational function has a vertical asymptote for any \( x \) value where the denominator is zero but the numerator is non-zero.
**Where Are Vertical Asymptotes?**

Where the function has an **infinite limit**.

- There may be no vertical asymptotes, or many.
- A rational function has a vertical asymptote for any $x$ value where the denominator is zero but the numerator is non-zero.
- Some functions have vertical asymptotes without having an obvious denominator to make equal to zero.
Where Are Vertical Asymptotes?

Where the function has an **infinite limit**.

- There may be no vertical asymptotes, or many.
- A rational function has a vertical asymptote for any $x$ value where the denominator is zero but the numerator is non-zero.
- Some functions have vertical asymptotes without having an obvious denominator to make equal to zero. For example, $\tan x$ has vertical asymptotes at any odd multiple of $\frac{\pi}{2}$.
Horizontal Asymptotes and Limits at Infinity
Horizontal Asymptotes and Limits at Infinity

What Are the Horizontal Asymptotes?
Horizontal Asymptotes and Limits at Infinity

What Are the Horizontal Asymptotes?

- To identify any horizontal asymptotes for a function determine the **limits at infinity**.
What Are the Horizontal Asymptotes?

- To identify any horizontal asymptotes for a function determine the **limits at infinity**.

- There can be at most two different horizontal asymptotes: on the right \((x \to \infty)\) and on the left \((x \to -\infty)\).
Horizontal Asymptotes and Limits at Infinity

What Are the Horizontal Asymptotes?

- To identify any horizontal asymptotes for a function determine the limits at infinity.
- There can be at most two different horizontal asymptotes: on the right ($x \to \infty$) and on the left ($x \to -\infty$).
- A rational function can have only one horizontal asymptote.
What Are the Horizontal Asymptotes?

- To identify any horizontal asymptotes for a function determine the **limits at infinity**.
- There can be at most two different horizontal asymptotes: on the right \((x \to \infty)\) and on the left \((x \to -\infty)\).
- A rational function can have only one horizontal asymptote.
  - If the degree of the denominator and numerator are equal, there is a **non-zero** horizontal asymptote.
Horizontal Asymptotes and Limits at Infinity

What Are the Horizontal Asymptotes?

- To identify any horizontal asymptotes for a function determine the limits at infinity.
- There can be at most two different horizontal asymptotes: on the right \((x \to \infty)\) and on the left \((x \to -\infty)\).
- A rational function can have only one horizontal asymptote.
  - If the degree of the denominator and numerator are equal, there is a **non-zero** horizontal asymptote.
  - If the degree of the denominator is **greater than** the degree of the numerator, there is a horizontal asymptote at \(y = 0\).
What Are the Horizontal Asymptotes?

- To identify any horizontal asymptotes for a function determine the **limits at infinity**.
- There can be at most two different horizontal asymptotes: on the right \((x \to \infty)\) and on the left \((x \to -\infty)\).
- A rational function can have only one horizontal asymptote.
  - If the degree of the denominator and numerator are equal, there is a **non-zero** horizontal asymptote.
  - If the degree of the denominator is **greater than** the degree of the numerator, there is a horizontal asymptote at \(y = 0\).
  - If the degree of the denominator is **less than** the degree of the numerator, there is no horizontal asymptote.
What Are the Horizontal Asymptotes?

- To identify any horizontal asymptotes for a function determine the limits at infinity.
- There can be at most two different horizontal asymptotes: on the right \((x \to \infty)\) and on the left \((x \to -\infty)\).
- A rational function can have only one horizontal asymptote.
  - If the degree of the denominator and numerator are equal, there is a non-zero horizontal asymptote.
  - If the degree of the denominator is greater than the degree of the numerator, there is a horizontal asymptote at \(y = 0\).
  - If the degree of the denominator is less than the degree of the numerator, there is no horizontal asymptote.
- Functions involving exponential functions may have different horizontal asymptotes on the left and the right.
Example 21 – Investigating Asymptotes

For each function below, determine any vertical and horizontal asymptotes and sketch the graph of the function. For the vertical asymptote(s) determine the behavior of the function on either side of the asymptote by calculating the appropriate one-sided infinite limits.

(a) \( f(x) = \frac{1}{x + 1} \)

(b) \( g(x) = \frac{x}{x + 1} \)

(c) \( h(x) = \frac{1}{(x + 1)^2} \)

(d) \( F(x) = \frac{x}{(x + 1)^2} \)

(e) \( G(x) = \frac{x^2}{(x + 1)^2} \)

(f) \( H(x) = \frac{x^3}{(x + 1)^2} \)
Solution: Example 21(a)

For

\[ f(x) = \frac{1}{x + 1} \]

there is a vertical asymptote where
Solution: Example 21(a)

For

\[ f(x) = \frac{1}{x+1} \]

there is a vertical asymptote where

\[ x + 1 = 0 \]
Example 21 – Investigating Asymptotes

Solution: Example 21(a)

For

\[ f(x) = \frac{1}{x + 1} \]

there is a vertical asymptote where

\[ x + 1 = 0 \Rightarrow x = -1 \]
Solution: Example 21(a)

For

\[ f(x) = \frac{1}{x+1} \]

there is a vertical asymptote where

\[ x + 1 = 0 \Rightarrow x = -1 \]

The behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1^-} \frac{1}{x + 1} \]
Solution: Example 21(a)

For 

\[ f(x) = \frac{1}{x + 1} \]

there is a vertical asymptote where 

\[ x + 1 = 0 \Rightarrow x = -1 \]

The behavior on either side of the vertical asymptote at \( x = -1 \) is given by 

\[ \lim_{{x \to -1^-}} \frac{1}{x + 1} = -\infty \]
Solution: Example 21(a)

For

\[ f(x) = \frac{1}{x + 1} \]

there is a vertical asymptote where

\[ x + 1 = 0 \Rightarrow x = -1 \]

The behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1^-} \frac{1}{x + 1} = -\infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{1}{x + 1} \]
Example 21 – Investigating Asymptotes

Solution: Example 21(a)

For

\[ f(x) = \frac{1}{x+1} \]

there is a vertical asymptote where

\[ x + 1 = 0 \Rightarrow x = -1 \]

The behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1^-} \frac{1}{x+1} = -\infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{1}{x+1} = \infty \]
Solution: Example 21(a)

For

\[ f(x) = \frac{1}{x + 1} \]

there is a vertical asymptote where

\[ x + 1 = 0 \implies x = -1 \]

The behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1^-} \frac{1}{x + 1} = -\infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{1}{x + 1} = \infty \]

Further,

\[ \lim_{x \to \pm\infty} \frac{1}{x + 1} \]
Solution: Example 21(a)

For

\[ f(x) = \frac{1}{x + 1} \]

there is a vertical asymptote where

\[ x + 1 = 0 \Rightarrow x = -1 \]

The behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1^-} \frac{1}{x + 1} = -\infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{1}{x + 1} = \infty \]

Further,

\[ \lim_{x \to \pm\infty} \frac{1}{x + 1} = 0 \]
Solution: Example 21(a)

For

\[ f(x) = \frac{1}{x + 1} \]

there is a vertical asymptote where

\[ x + 1 = 0 \Rightarrow x = -1 \]

The behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1^-} \frac{1}{x + 1} = -\infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{1}{x + 1} = \infty \]

Further,

\[ \lim_{x \to \pm \infty} \frac{1}{x + 1} = 0 \]

So that, the \( x \)-axis, \( y = 0 \), is the horizontal asymptote for the graph of the function.
Solution: Example 21(a)

For

\[ f(x) = \frac{1}{x + 1} \]

there is a vertical asymptote where

\[ x + 1 = 0 \Rightarrow x = -1 \]

The behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1^-} \frac{1}{x + 1} = -\infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{1}{x + 1} = \infty \]

Further,

\[ \lim_{x \to \pm\infty} \frac{1}{x + 1} = 0 \]

So that, the \( x \)-axis, \( y = 0 \), is the horizontal asymptote for the graph of the function.
Solution: Example 21(b)

For

\[ g(x) = \frac{x}{x + 1} \]

again there is a vertical asymptote at
Solution: Example 21(b)

For

$$g(x) = \frac{x}{x + 1}$$

again there is a vertical asymptote at $x = -1$. 

Clint Lee  
Math 112 Lecture 5: Limits at Infinity and Asymptotes
Solution: Example 21(b)

For

\[ g(x) = \frac{x}{x + 1} \]

again there is a vertical asymptote at \( x = -1 \). Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1^-} \frac{x}{x + 1} \]
Solution: Example 21(b)

For

\[ g(x) = \frac{x}{x + 1} \]

again there is a vertical asymptote at \( x = -1 \).

Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{{x \to -1^-}} \frac{x}{x + 1} = \infty \]
Solution: Example 21(b)

For

\[ g(x) = \frac{x}{x + 1} \]

again there is a vertical asymptote at \( x = -1 \).
Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1^-} \frac{x}{x + 1} = \infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{x}{x + 1} \]
Solution: Example 21(b)

For

\[ g(x) = \frac{x}{x + 1} \]

again there is a vertical asymptote at \( x = -1 \). Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[
\lim_{x \to -1^-} \frac{x}{x + 1} = \infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{x}{x + 1} = -\infty
\]
Solution: Example 21(b)

For

\[ g(x) = \frac{x}{x + 1} \]

g again there is a vertical asymptote at \( x = -1 \).
Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1^-} \frac{x}{x + 1} = \infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{x}{x + 1} = -\infty \]

Further,

\[ \lim_{x \to \pm\infty} \frac{x}{x + 1} \]
Solution: Example 21(b)

For

\[ g(x) = \frac{x}{x+1} \]

again there is a vertical asymptote at \( x = -1 \).

Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1^-} \frac{x}{x+1} = \infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{x}{x+1} = -\infty \]

Further,

\[ \lim_{x \to \pm \infty} \frac{x}{x+1} = 1 \]
Solution: Example 21(b)

For

\[ g(x) = \frac{x}{x + 1} \]

again there is a vertical asymptote at \( x = -1 \).

Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1^-} \frac{x}{x + 1} = \infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{x}{x + 1} = -\infty \]

Further,

\[ \lim_{x \to \pm \infty} \frac{x}{x + 1} = 1 \]

So that \( y = 1 \) is the horizontal asymptote for the graph of the function.
Solution: Example 21(b)

For
\[ g(x) = \frac{x}{x + 1} \]
again there is a vertical asymptote at \( x = -1 \).
Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[
\lim_{x \to -1^-} \frac{x}{x + 1} = \infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{x}{x + 1} = -\infty
\]

Further,

\[
\lim_{x \to \pm \infty} \frac{x}{x + 1} = 1
\]

So that \( y = 1 \) is the horizontal asymptote for the graph of the function.
Solution: Example 21(c)

For

\[ h(x) = \frac{1}{(x + 1)^2} \]

there is still a vertical asymptote at
Solution: Example 21(c)

For

\[ h(x) = \frac{1}{(x+1)^2} \]

there is still a vertical asymptote at \( x = -1 \).
Solution: Example 21(c)

For

\[ h(x) = \frac{1}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).

Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1^-} \frac{1}{(x + 1)^2} \]
Solution: Example 21(c)

For

\[ h(x) = \frac{1}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).
Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1^-} \frac{1}{(x + 1)^2} = \infty \]
Solution: Example 21(c)

For

\[ h(x) = \frac{1}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \). Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[
\lim_{x \to -1^-} \frac{1}{(x + 1)^2} = \infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{1}{(x + 1)^2}
\]
Solution: Example 21(c)

For

\[ h(x) = \frac{1}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).

Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1^-} \frac{1}{(x + 1)^2} = \infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{1}{(x + 1)^2} = \infty \]
Example 21 – Investigating Asymptotes

Solution: Example 21(c)

For

\[ h(x) = \frac{1}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \). Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1^-} \frac{1}{(x + 1)^2} = \infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{1}{(x + 1)^2} = \infty \]

Further,

\[ \lim_{x \to \pm \infty} \frac{1}{(x + 1)^2} \]
Solution: Example 21(c)

For

$$h(x) = \frac{1}{(x + 1)^2}$$

there is still a vertical asymptote at $x = -1$. 

Now the behavior on either side of the vertical asymptote at $x = -1$ is given by

$$\lim_{x \to -1^-} \frac{1}{(x + 1)^2} = \infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{1}{(x + 1)^2} = \infty$$

Further,

$$\lim_{x \to \pm \infty} \frac{1}{(x + 1)^2} = 0$$
Solution: Example 21(c)

For

\[ h(x) = \frac{1}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).

Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[
\lim_{{x \to -1^-}} \frac{1}{(x + 1)^2} = \infty \quad \text{and} \quad \lim_{{x \to -1^+}} \frac{1}{(x + 1)^2} = \infty
\]

Further,

\[
\lim_{{x \to \pm \infty}} \frac{1}{(x + 1)^2} = 0
\]

So that \( y = 0 \) is the horizontal asymptote for the graph of the function.
Solution: Example 21(c)

For

$$h(x) = \frac{1}{(x + 1)^2}$$

there is still a vertical asymptote at $x = -1$. Now the behavior on either side of the vertical asymptote at $x = -1$ is given by

$$\lim_{x \to -1^-} \frac{1}{(x + 1)^2} = \infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{1}{(x + 1)^2} = \infty$$

Further,

$$\lim_{x \to \pm \infty} \frac{1}{(x + 1)^2} = 0$$

So that $y = 0$ is the horizontal asymptote for the graph of the function.
Solution: Example 21(d)

For

\[ F(x) = \frac{x}{(x + 1)^2} \]

there is still a vertical asymptote at
Solution: Example 21(d)

For

\[ F(x) = \frac{x}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).
Solution: Example 21(d)

For

\[ F(x) = \frac{x}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).

Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1} \frac{x}{(x + 1)^2} \]
Solution: Example 21(d)

For

\[ F(x) = \frac{x}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \). Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[
\lim_{x \to -1} \frac{x}{(x + 1)^2} = -\infty
\]
Solution: Example 21(d)

For

\[ F(x) = \frac{x}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).

Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1} \frac{x}{(x + 1)^2} = -\infty \]

Further,

\[ \lim_{x \to \pm\infty} \frac{x}{(x + 1)^2} \]
Solution: Example 21(d)

For

\[ F(x) = \frac{x}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \). Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1} \frac{x}{(x + 1)^2} = -\infty \]

Further,

\[ \lim_{x \to \pm\infty} \frac{x}{(x + 1)^2} = 0 \]
Solution: Example 21(d)

For

\[ F(x) = \frac{x}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).

Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1} \frac{x}{(x + 1)^2} = -\infty \]

Further,

\[ \lim_{x \to \pm\infty} \frac{x}{(x + 1)^2} = 0 \]

So that \( y = 0 \) is the horizontal asymptote for the graph of the function.
Solution: Example 21(d)

For

\[ F(x) = \frac{x}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).

Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1} \frac{x}{(x + 1)^2} = -\infty \]

Further,

\[ \lim_{x \to \pm\infty} \frac{x}{(x + 1)^2} = 0 \]

So that \( y = 0 \) is the horizontal asymptote for the graph of the function.
Solution: Example 21(e)

For

\[ G(x) = \frac{x^2}{(x + 1)^2} \]

there is still a vertical asymptote at
Solution: Example 21(e)

For

\[ G(x) = \frac{x^2}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).
Solution: Example 21(e)

For

\[ G(x) = \frac{x^2}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).

Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1} \frac{x^2}{(x + 1)^2} = \]
Example 21 – Investigating Asymptotes

Solution: Example 21(e)

For

\[ G(x) = \frac{x^2}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).

Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1} \frac{x^2}{(x + 1)^2} = \infty \]
Solution: Example 21(e)

For

\[ G(x) = \frac{x^2}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).

Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1} \frac{x^2}{(x + 1)^2} = \infty \]

Further,

\[ \lim_{x \to \pm \infty} \frac{x^2}{(x + 1)^2} \]
Solution: Example 21(e)

For

\[ G(x) = \frac{x^2}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).
Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1} \frac{x^2}{(x + 1)^2} = \infty \]

Further,

\[ \lim_{x \to \pm\infty} \frac{x^2}{(x + 1)^2} = 1 \]
Solution: Example 21(e)

For

\[ G(x) = \frac{x^2}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).

Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1} \frac{x^2}{(x + 1)^2} = \infty \]

Further,

\[ \lim_{x \to \pm \infty} \frac{x^2}{(x + 1)^2} = 1 \]

So that \( y = 1 \) is the horizontal asymptote for the graph of the function.
Solution: Example 21(e)

For

\[ G(x) = \frac{x^2}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).

Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1} \frac{x^2}{(x + 1)^2} = \infty \]

Further,

\[ \lim_{x \to \pm\infty} \frac{x^2}{(x + 1)^2} = 1 \]

So that \( y = 1 \) is the horizontal asymptote for the graph of the function.
Solution: Example 21(f)

For

\[ H(x) = \frac{x^3}{(x + 1)^2} \]

there is still a vertical asymptote at
Solution: Example 21(f)

For

\[ H(x) = \frac{x^3}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).
Solution: Example 21(f)

For

\[ H(x) = \frac{x^3}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).

Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1} \frac{x^3}{(x + 1)^2} = \]
Example 21 – Investigating Asymptotes

Solution: Example 21(f)

For

\[ H(x) = \frac{x^3}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \).

Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{{x \to -1}} \frac{x^3}{(x + 1)^2} = -\infty \]
Solution: Example 21(f)

For

\[ H(x) = \frac{x^3}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \). Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1} \frac{x^3}{(x + 1)^2} = -\infty \]

Further,

\[ \lim_{x \to \pm\infty} \frac{x^3}{(x + 1)^2} \]
Solution: Example 21(f)

For

\[ H(x) = \frac{x^3}{(x + 1)^2} \]

there is still a vertical asymptote at \( x = -1 \). Now the behavior on either side of the vertical asymptote at \( x = -1 \) is given by

\[ \lim_{x \to -1} \frac{x^3}{(x + 1)^2} = -\infty \]

Further,

\[ \lim_{x \to \pm\infty} \frac{x^3}{(x + 1)^2} = \pm\infty \]
Solution: Example 21(f)

For

$$H(x) = \frac{x^3}{(x + 1)^2}$$

there is still a vertical asymptote at $x = -1$. Now the behavior on either side of the vertical asymptote at $x = -1$ is given by

$$\lim_{x \to -1} \frac{x^3}{(x + 1)^2} = -\infty$$

Further,

$$\lim_{x \to \pm \infty} \frac{x^3}{(x + 1)^2} = \pm \infty$$

So there is no horizontal asymptote.
Solution: Example 21(f)

For

$$H(x) = \frac{x^3}{(x + 1)^2}$$

there is still a vertical asymptote at $x = -1$. Now the behavior on either side of the vertical asymptote at $x = -1$ is given by

$$\lim_{x \to -1} \frac{x^3}{(x + 1)^2} = -\infty$$

Further,

$$\lim_{x \to \pm \infty} \frac{x^3}{(x + 1)^2} = \pm \infty$$

So there is no horizontal asymptote.
Solution: Example 21(f) continued

Note that

\[
\frac{x^3}{(x + 1)^2}
\]
Solution: Example 21(f) continued

Note that

\[
\frac{x^3}{(x + 1)^2} = \frac{x^3 + 2x^2 + x - 2x^2 - 4x - 2 + 3x + 2}{(x + 1)^2}
\]
Solution: Example 21(f) continued

Note that

\[
\frac{x^3}{(x+1)^2} = \frac{x^3 + 2x^2 + x - 2x^2 - 4x - 2 + 3x + 2}{(x+1)^2} = \frac{x(x+1)^2 - 2(x+1)^2 + 3x + 2}{(x+1)^2}
\]
Solution: Example 21(f) continued

Note that

\[
\frac{x^3}{(x + 1)^2} = \frac{x^3 + 2x^2 + x - 2x^2 - 4x - 2 + 3x + 2}{(x + 1)^2} = \frac{x(x + 1)^2 - 2(x + 1)^2 + 3x + 2}{(x + 1)^2} = x - 2 + \frac{3x + 2}{(x + 1)^2}
\]
Example 21 – Investigating Asymptotes

Solution: Example 21(f) continued

Note that

\[
\frac{x^3}{(x+1)^2} = \frac{x^3 + 2x^2 + x - 2x^2 - 4x - 2 + 3x + 2}{(x+1)^2}
\]

\[
= \frac{\cancel{x^3} (x+1)^2 - 2\, \cancel{(x+1)^2} + \cancel{3x} + \cancel{2}}{(x+1)^2}
\]

\[
= x - 2 + \frac{3x + 2}{(x+1)^2}
\]

Now

\[
\lim_{x \to \pm \infty} \frac{3x + 2}{(x+1)^2} = 0
\]

so we say that as \( x \to \pm \infty \) the function \( H(x) \) behaves like \( x - 2 \). We call the line \( y = x - 2 \) a **slant asymptote**.