Student Name: ____________________

Student Number: ____________________

Total Marks: ________
100

Okanagan University College
Final Examination

Math 122 (Winter, 2003)

Instructor(s): Clint Lee

Section(s): 71 & 72

April 17, 2003

9:00AM

Duration: 3 hours

READ INSTRUCTIONS CAREFULLY BEFORE COMMENCING EXAM

INSTRUCTIONS: Answer all 13 questions in the spaces provided, showing all significant steps. Partial marks will be awarded for correct work even if the final answer is incorrect. Marks per question are given in the left margin, total 100. Check that your paper contains all 13 pages in addition to the cover page. The last page of the examination is a formula sheet. You may detach this sheet and use it in any of the problems on the exam.

This paper contains pages numbered 1 to 13

EXAM BOOKLETS ARE NOT REQUIRED

1 Evaluate each integral. Give the exact numerical value of any definite integral.

[3] (a) $\int \left(\frac{1}{x^2} + \frac{3}{3x+1} - e^{-x/2} \right) dx$

$$\begin{aligned} \int \left(\frac{1}{x^2} + \frac{3}{3x+1} - e^{-x/2} \right) dx &= \int \left(x^{-2} + \frac{3}{3x+1} - e^{-x/2} \right) dx \\ &= -\frac{1}{x} + \ln|3x+1| + 2e^{-x/2} + C \end{aligned}$$

[3] (b) $\int_1^2 \frac{t^2}{\sqrt[3]{9-t^3}}\, dt$

Make the substitution $u = 9 = t^3$ so that $du = -3t^2\, dt$ or $t^2\, dt = -\frac{1}{3}\, du$. Further, $t = 2 \Rightarrow u = 1$ and $t = 1 \Rightarrow u = 8$. Then,

$$\int_1^2 \frac{t^2}{\sqrt{9-t^3}}\, dt = -\frac{1}{3} \int_8^1 u^{-1/3}\, du = \frac{1}{3}\frac{3}{2} u^{2/3} \Big|_1^8 = \frac{3}{2}$$

[3] (c) $\int_0^{\pi/4} w^2 \cos 2w\, dw$

Use integration by parts with

$$u = w^2 \qquad dv = \cos 2w\, dw$$
$$du = 2w\, dw \qquad v = \frac{1}{2}\sin 2w$$

Then

$$\begin{aligned} \int_0^{\pi/4} w^2 \cos 2w\, dw &= \frac{1}{2} w^2 \sin 2w \Big|_0^{\pi/4} - \int_0^{\pi/4} w \sin 2w\, dw \\ &= \frac{\pi^2}{32} - \int_0^{\pi/4} w \sin 2w\, dw \end{aligned}$$

Integration by parts again with

$$u = w \qquad dv = \sin 2w\, dw$$
$$du = dw \qquad v = -\frac{1}{2}\cos 2w$$

Then

$$\begin{aligned} \int_0^{\pi/4} w^2 \cos 2w\, dw &= \frac{\pi^2}{32} - \left(-\frac{1}{2} w \sin 2w \Big|_0^{\pi/4} + \frac{1}{2} \int_0^{\pi/4} \cos 2w\, dw \right) \\ &= \frac{\pi^2}{32} - \frac{1}{4} \sin 2w \Big|_0^{\pi/4} = \frac{\pi^2}{32} - \frac{1}{4} \end{aligned}$$

[3] (d) $\int \frac{x^2+x-1}{(x+2)(x-3)}\, dx$

Expand the denominator to give $(x+2)(x-3) = x^2 - x - 6$. Then use polynomial long division to write

$$\frac{x^2+x-1}{(x+2)(x-3)} = 1 + \frac{2x+5}{(x+2)(x-3)}$$

Then

$$\begin{aligned} \int \frac{x^2+x-1}{(x+2)(x-3)}\, dx &= \int \left(1 + \frac{2x+5}{(x+2)(x-3)} \right) dx \\ &= x + \int \left(\frac{2x+5}{(x+2)(x-3)} \right) dx \end{aligned}$$

Next use partial fractions to write

$$\frac{2x+5}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$$

where A and B satisfy $2x+5 = A(x-3) + B(x+2)$. Plugging in $x = -2$ and $x = 3$ gives $A = -\frac{1}{5}$ and $B = \frac{11}{5}$. Thus,

$$\int \frac{x^2+x-1}{(x+2)(x-3)}\, dx = x + \frac{1}{5} \int \left(-\frac{1}{x+2} + \frac{11}{x-3} \right) dx = x - \frac{1}{5} \ln|x+2| + \frac{11}{5} \ln|x-3| + C$$

2 During the first eight weeks after she is born a puppy grows at a rate $r(t)$, in kilograms per week, as shown in the graph at the right. The puppy weighed 0.5 kg when she was born.

[2] (a) Set up a definite integral giving $W(t)$, the puppy's weight t weeks after she was born.

$$W(t) - W(0) = \int_0^t r(z)\,dz$$

Since $W(0) = 0.5$, this gives

$$W(t) = 0.5 + \int_0^t r(z)\,dz$$

[2] (b) Determine the puppy's weight

i) 2 weeks after she was born

The area represented by the integral here is a trapezoid, so

$$W(2) = 0.5 + \int_0^2 r(z)\,dz = 0.5 + \left(\frac{0.5 + 1.0}{2}\right)(2) = 2.0\,\text{kg}$$

ii) 4 weeks after she was born

Taking the result from the previous part as a starting point, we have

$$W(4) = 2.0 + \int_2^4 r(z)\,dz = 2.0 + \left(\frac{1.0 + 2.0}{2}\right)(2) = 5.0\,\text{kg}$$

iii) 6 weeks after she was born

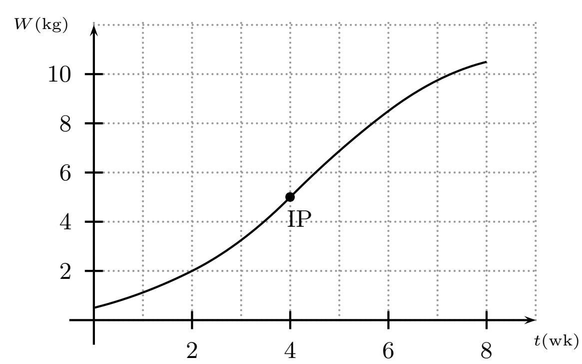
$$W(6) = 5.0 + \int_4^6 r(z)\,dz = 5.0 + \left(\frac{2.0 + 1.5}{2}\right)(2) = 8.5\,\text{kg}$$

iv) 8 weeks after she was born

$$W(8) = 8.5 + \int_6^8 r(z)\,dz = 8.5 + \left(\frac{1.5 + 0.5}{2}\right)(2) = 10.5\,\text{kg}$$

[2] (c) Sketch a graph of $W(t)$ over the first 8 weeks after the puppy was born. Show the intervals where $W(t)$ is increasing and decreasing, the absolute maximum and minimum values, and the inflection point(s) of the graph.

The function $W(t)$ is always increasing since $W'(t) = r(t) > 0$ for all t. The inflection point is at the point where $r(t)$ is maximum, which is at $t = 4$. At this time the puppy's weight is $W(4) = 5$.



[2] 3 (a) Write the definite integral $\int_{-1}^{2} \frac{e^x}{x+2}\,dx$ as a limit of the Riemann sum for the integral using right endpoints.

Here

$$\Delta x = \frac{3}{n} \quad \text{and} \quad x_i = -1 + \frac{3i}{n}$$

Thus, the right endpoint Riemann sum for the integral is

$$\begin{aligned}\int_{-1}^{2} \frac{e^x}{x+2}\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{e^{-1+\frac{3i}{n}}}{-1+\frac{3i}{n}+2}\right)\left(\frac{3}{n}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{e^{-1+\frac{3i}{n}}}{1+\frac{3i}{n}}\right)\left(\frac{3}{n}\right)\end{aligned}$$

[2] (b) Write a definite integral given by the limit: $\lim_{n\to\infty}\sum_{i=1}^{n}\tan\left(\frac{\pi}{4}+\frac{\pi i}{12n}\right)\left(\frac{\pi}{12n}\right)$

Here

$$\Delta x = \frac{\pi}{12n} \quad \text{and} \quad a = x_0 = \frac{\pi}{4}, b = x_n = \frac{\pi}{4} + \frac{\pi}{12} = \frac{\pi}{3}$$

or equivalently

$$a = x_0 = 0, b = x_n = \frac{\pi}{12}$$

So two possible definite integrals are

$$\lim_{n\to\infty}\sum_{i=1}^{n}\tan\left(\frac{\pi}{4}+\frac{\pi i}{12n}\right)\left(\frac{\pi}{12n}\right) = \int_{\pi/4}^{\pi/3}\tan x\,dx = \int_{0}^{\pi/12}\tan\left(\frac{\pi}{4}+x\right)\,dx$$

4 The Q **integral** function is defined as

$$\mathrm{Q}(x) = \int_0^x te^{-\cos t}\, dt$$

[2] (a) Find $\dfrac{d}{dx}\,\mathrm{Q}(x)$.

Using the Fundamental Theorem of Calculus, Part 1, we have

$$\frac{dQ}{dx} = \frac{d}{dx}\int_0^x te^{-\cos t}\, dt = xe^{-\cos x}$$

[3] (b) Verify that

$$\frac{d}{dx}\,\mathrm{Q}(x)e^{\cos x} = -\sin x e^{\cos x}\,\mathrm{Q}(x) + x$$

Using the product rule together with the result from part (a) above we have

$$\begin{aligned}\frac{d}{dx}\,\mathrm{Q}(x)e^{\cos x} &= \frac{dQ}{dx}e^{\cos x} + Qe^{\cos x}\,(-\sin x)\\ &= \left(xe^{-\cos x}\right)e^{\cos x} - Qe^{\cos x}\sin x = x - Q(x)e^{\cos x}\sin x\end{aligned}$$

[2] (c) Verify that the function $y = Q(x)e^{\cos x} + Ce^{\cos x}$, where C is an arbitrary constant, is the general solution to the differential equation

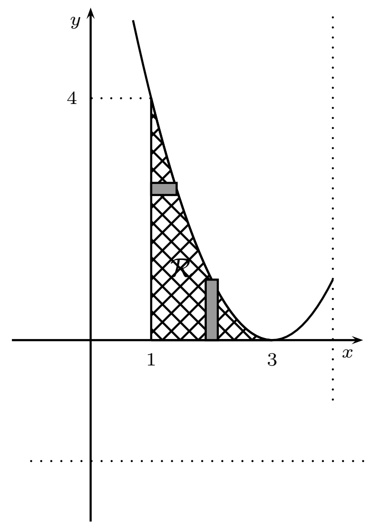
$$\frac{dy}{dx} + (\sin x)\, y = x$$

From the result in part (b) above we have

$$\frac{dy}{dx} = x - Q(x)e^{\cos x}\sin x - Ce^{\cos x}\sin x$$

so that

$$\begin{aligned}\frac{dy}{dx} + (\sin x)\, y &= x - Q(x)e^{\cos x}\sin x - Ce^{\cos x}\sin x + \sin x\left(Q(x)e^{\cos x} + Ce^{\cos x}\right)\\ &= x - Q(x)e^{\cos x}\sin x - Ce^{\cos x}\sin x + Q(x)e^{\cos x}\sin x + Ce^{\cos x}\sin x\\ &= x\end{aligned}$$



5 The diagram shows the first quadrant region $\mathcal{R}$ bounded by $y = (x-3)^2$, $y = 0$, and $x = 1$.

[3] (a) Use the disc method to find the volume of the solid Then generated when the region $\mathcal{R}$ is rotated about the x-axis.

Using the disc method for rotation about the x-axis means we use vertical rectangles and integrate in x. Then

$$\Delta V = \pi r^2 \Delta x = \pi\left[(x-3)^2\right]\Delta x = \pi(x-3)^4 \Delta x$$

$$V = \pi \int_1^3 (x-3)^4\,dx = \frac{\pi}{5}(x-3)^5\bigg|_1^3 = \frac{\pi}{5}\left(0-(-2)^5\right) = \frac{32\pi}{5}$$

[3] (b) Use the cylindrical shell method to find the volume of the solid generated when the region $\mathcal{R}$ is rotated about the y-axis.

Using the cylindrical shell method for rotation about the y-axis means we use vertical rectangles and integrate in x. Then

$$\Delta V = 2\pi r h \Delta x = 2\pi x(x-3)^2\Delta x = 2\pi x\left(x^2-6x+9\right)\Delta x = 2\pi\left(x^3-6x^2+9x\right)\Delta x$$

Then

$$V = 2\pi\int_1^3\left(x^3-6x^2+9x\right)\,dx = 2\pi\left(\frac{x^4}{4}-2x^3+\frac{9x^2}{2}\right)\bigg|_1^3$$
$$= 2\pi\left(\frac{81}{4}-54+\frac{81}{2}-\frac{1}{4}+2-\frac{9}{2}\right) = 8\pi$$

[3] (c) Set up the integral to use the washer method to find the volume of the solid generated when the region $\mathcal{R}$ is rotated about the line $x = 4$. **Do not** simplify the integrand or evaluate the integral.

Using the washer method for rotation about a vertical line means we use horizontal rectangles and integrate in y. First solve for x in terms of y in the equation for the curve.

$$y = (x-3)^2 \Rightarrow x = 3 \pm \sqrt{y}$$

Since the portion of the parabola we are using is the left half, we take the minus in this expression. Thus, $x = 3-\sqrt{y}$. Then

$$\Delta V = \pi\left(r_o^2 - r_i^2\right)\Delta y = \pi\left[(4-1)^2-(4-(3-\sqrt{y}))^2\right]\Delta y$$
$$= \pi\left[9-(1+\sqrt{y})^2\right]\Delta y$$

So the volume is

$$V = \pi\int_0^4\left[9-(1+\sqrt{y})^2\right]\,dy$$

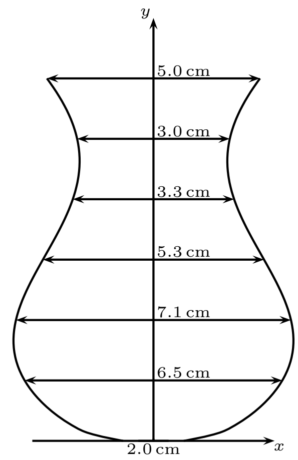
[2] (d) Set up the integral to use the cylindrical shell method to find the volume of the solid generated when the region $\mathcal{R}$ is rotated about the line $y = -2$. **Do not** simplify the integrand or evaluate the integral.

Using the cylindrical shell method for rotation about a horizontal axis means that we use horizontal rectangles and integrate in y. We solved for x in terms of y in part (c) above. Then

$$\Delta V = 2\pi r h \Delta y = 2\pi(y+2)\left(3-\sqrt{y}-1\right)\Delta y = 2\pi(y+2)\left(2-\sqrt{y}\right)\Delta y$$

So the volume is

$$V = 2\pi\int_0^4 (y+2)\left(2-\sqrt{y}\right)\,dy$$



6 Jordan has bought a new vase and she wants to compute the volume of water that the vase holds. She measures the height of the vase to be 12 cm and diameter of the vase at 2 cm intervals. The diameter measurements are shown in the diagram. Note that the diameter at the base is 2 cm.

[2] (a) Let $d(y)$ be the diameter of the vase as a function of the distance y from its bottom. Set up an integral for the volume of the vase in terms of the function $d(y)$.

The radius of each circular cross-section is $r = \dfrac{d(y)}{2}$, so that the volume is

$$V = \pi \int_0^{12} \left[\frac{d(y)}{2}\right]^2 \, dy = \frac{\pi}{4} \int_0^{12} [d(y)]^2 \, dy$$

[4] (b) Use Simpson's rule to estimate value of the integral in part (a).

Here $n = 6$ and $\Delta y = 2$. Then

$$\begin{aligned} V \approx S_6 &= \frac{\pi}{4} \cdot \frac{2}{3} \left([d(0)]^2 + 4\,[d(2)]^2 + 2\,[d(4)]^2 + 4\,[d(6)]^2 + 2\,[d(8)]^2 + 4\,[d(10)]^2 + [d(12)]^2 \right) \\ &= \frac{\pi}{6} \left(2^2 + 4(6.5)^2 + 2(7.1)^2 + 4(5.3)^2 + 2(3.3)^2 + 4(3.0)^2 + (5.0)^2 \right) \\ &= \frac{\pi}{6}(468.96) = 245.5\,\text{cm}^3 \end{aligned}$$

[3] 7 Use the Comparison Test to determine whether the integral below converges or diverges.

$$\int_0^\infty \frac{e^{-x}}{1+\sqrt{x}} \, dx$$

Hint: Explain why $1 + \sqrt{x} \geq 1$ for all $x \geq 0$.

For $x \geq 0$, we have $\sqrt{x} \geq 0$. Adding 1 to both sides gives $1 + \sqrt{x} \geq 1$. Thus,

$$\frac{1}{1+\sqrt{x}} \leq 1 \Rightarrow \frac{e^{-x}}{1+\sqrt{x}} \leq e^{-x}$$

Further,

$$\begin{aligned} \int_0^\infty e^{-x} \, dx &= \lim_{t\to\infty} \int_0^t e^{-x} \, dx = \lim_{t\to\infty} \left(-e^{-x}\right)\Big|_0^t \\ &= \lim_{t\to\infty} \left(1 - e^{-t}\right) = 1 \end{aligned}$$

So the improper integral $\displaystyle\int_0^\infty e^{-x} \, dx$ converges. Hence, since $\dfrac{e^{-x}}{1+\sqrt{x}} \leq e^{-x}$, the Comparison Test shows that $\displaystyle\int_0^\infty \frac{e^{-x}}{1+\sqrt{x}} \, dx$ converges.

[3] 8 (a) Use an appropriate trigonometric substitution to evaluate the integral: $\displaystyle\int \frac{x^2}{(x^2+4)^{3/2}}\,dx$

Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta d\theta$ and $\left(x^2+4\right)^{3/2} = 8\sec^3\theta$. Then

$$\int \frac{x^2}{(x^2+4)^{3/2}}\,dx = \int \frac{4\tan^2\theta}{8\sec^3\theta}\left(2\sec^2\theta d\theta\right) = \int \frac{\tan^2\theta}{\sec\theta}\,d\theta = \int \left(\frac{\sec^2\theta - 1}{\sec\theta}\right)\,d\theta$$
$$= \int (\sec\theta - \cos\theta)\,d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta + C$$

Now, $\tan\theta = \dfrac{x}{3}$ so that $\sec\theta = \dfrac{\sqrt{x^2+4}}{2}$ and $\sin\theta = \dfrac{x}{\sqrt{x^2+4}}$. Thus,

$$\int \frac{x^2}{(x^2+4)^{3/2}}\,dx = \ln\left|\frac{\sqrt{x^2+4}+x}{2}\right| - \frac{x}{\sqrt{x^2+4}} + C = \ln\left|\sqrt{x^2+4}+x\right| - \frac{x}{\sqrt{x^2+4}} + C$$

[2] (b) Recall the hyperbolic functions $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$, and $\tanh x = \dfrac{\sinh x}{\cosh x}$. Further, recall that

$$\cosh^2 x - \sinh^2 x = 1, \quad \frac{d}{dx}\cosh x = \sinh x, \quad \frac{d}{dx}\sinh x = \cosh x$$

Use the definitions and identities above to show that

$$\frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x}$$

Using the definition of $\tanh x$ and the quotient rule

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{(\cosh x)(\cosh x) - (\sinh x)(\sinh x)}{(\cosh x)^2}$$
$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

[4] (c) Make the hyperbolic substitution $x = 2\sinh t$ in the integral in part (a) above and evaluate the resulting integral using the results in part (b) above. Express the result in terms of x using the fact that

$$\sinh^{-1} z = \ln\left|z + \sqrt{z^2+1}\right|$$

Letting $x = 2\sinh t$ gives $dx = 2\cosh t\,dt$. So that

$$\int \frac{x^2}{(x^2+4)^{3/2}}\,dx = \int \frac{4\sinh^2 t}{8\cosh^3 t}\,(2\cosh t\,dt)$$
$$= \int \frac{\sinh^2 t}{\cosh^2 t}\,dt = \int \tanh^2 t\,dt$$

Using the identities given in part (b) above we have

$$1 - \tanh^2 t = \frac{1}{\cosh^2 t} \Rightarrow \tanh^2 t = 1 - \frac{1}{\cosh^2 t}$$

Thus,

$$\int \frac{x^2}{(x^2+4)^{3/2}}\,dx = \int \left(1 - \frac{1}{\cosh^2 t}\right)\,dt$$
$$= t - \tanh t + C$$

Now, since $x = 2\sinh t$, we have

$$\sinh t = \frac{x}{2} \Rightarrow t = \sinh^{-1}\left(\frac{x}{2}\right)$$

and

$$\cosh^2 t = 1 + \sinh^2 t = 1 + \frac{x^2}{4} = \frac{4+x^2}{4} \Rightarrow \cosh t = \frac{1}{2}\sqrt{4+x^2}$$
$$\tanh t = \frac{\sinh t}{\cosh t} = \frac{x/2}{\sqrt{x^2+4}/2} = \frac{x}{\sqrt{4+x^2}}$$

So we have

$$\int \frac{x^2}{(x^2+4)^{3/2}}\,dx = \sinh^{-1}\left(\frac{x}{2}\right) - \frac{x}{\sqrt{4+x^2}} + C$$
$$= \ln\left|\frac{x}{2} + \sqrt{\frac{x^2}{4}+1}\right| - \frac{x}{\sqrt{4+x^2}} + C$$
$$= \ln\left|x + \sqrt{x^2+4}\right| - \frac{x}{\sqrt{4+x^2}} + C$$

[3] 9 (a) Use the substitution $z = x^2$ followed by integration by parts to evaluate the integral: $\int x^3 e^{-x^2}\,dx$

Letting $z = x^2$ gives $dz = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}\,dz$. Then

$$\int x^3 e^{-x^2}\,dx = \frac{1}{2}\int z e^{-z}\,dz$$

Now integration by parts with

$$u = z \qquad dv = e^{-z}\,dz$$
$$du = dz \qquad v = -e^{-z}$$

Then

$$\begin{aligned}\int x^3 e^{-x^2}\,dx &= \frac{1}{2}\left(-ze^{-z} + \int e^{-z}\,dz\right)\\ &= -\frac{1}{2}ze^{-z} - \frac{1}{2}e^{-z} + C\\ &= -\frac{1}{2}x^2e^{-x^2} - \frac{1}{2}e^{-x^2} + C\end{aligned}$$

[2] (b) Use the result in part (a) above to evaluate the improper integral: $\int_0^\infty x^3 e^{-x^2}\,dx$

Recall that $\lim_{x\to\infty} x^n e^{-x} = 0$ for any n.

$$\begin{aligned}\int_0^\infty x^3e^{-x^2}\,dx &= \lim_{t\to\infty}\int_0^t x^3e^{-x^2}\,dx\\ &= \lim_{t\to\infty}\left(-\frac{1}{2}x^2e^{-x^2} - \frac{1}{2}e^{-x^2}\right)\Bigg|_0^t\\ &= \lim_{t\to\infty}\left(\frac{1}{2} - \frac{1}{2}t^2e^{-t^2} - \frac{1}{2}e^{-t^2}\right)\\ &= \frac{1}{2}\end{aligned}$$

[3] (c) The *average speed* of molecules in an ideal gas is

$$\overline{v} = \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2}\int_0^\infty v^3 e^{-Mv^2/(2RT)}\,dv$$

where M is the molecular weight of the gas, R is the gas constant, T is the gas temperature, and v is molecular speed. Make the substitution $x = \sqrt{\frac{M}{2RT}}v$ in this integral and use the result in part (b) above to show that

$$\overline{v} = \sqrt{\frac{8RT}{\pi M}}$$

Letting $x = \sqrt{\frac{M}{2RT}}v$ gives $v = \sqrt{\frac{2RT}{M}}x \Rightarrow dv = \sqrt{\frac{2RT}{M}}\,dx$ and $\frac{Mv^2}{2RT} = x^2$. Thus

$$\begin{aligned}\overline{v} &= \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2}\int_0^\infty v^3e^{-Mv^2/(2RT)}\,dv\\ &= \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2}\left(\frac{2RT}{M}\right)^{3/2}\sqrt{\frac{2RT}{M}}\int_0^\infty x^3e^{-}x^2\,dx\\ &= \frac{4}{\sqrt{\pi}}\sqrt{\frac{2RT}{M}}\left(\frac{1}{2}\right) = \sqrt{\frac{4}{\pi}\frac{2RT}{M}} = \sqrt{\frac{8RT}{\pi M}}\end{aligned}$$

10 A second order chemical reaction involves the the interaction of one molecule of a reactant P with one molecule a second reactant Q to produce one molecule of the product X. This is written $P+Q \to X$. Let p and q be the initial concentrations of the reactants P and Q, respectively, and $x(t)$ be the concentration of the product X at time t. Then $p - x(t)$ and $q - x(t)$ are the concentrations of reactants P and Q at time t. The rate at which the product X is produced is proportional to the product of the concentrations of the two reactants P and Q.

[2] (a) Assuming that $p = q$, write a differential equation for $x(t)$, the concentration of the product X at time t. Let k be the constant of proportionality.

Since $p = q$ we can write the differential equation as

$$\frac{dx}{dt} = k(p-x)(q-x) = k(p-x)^2$$

[3] (b) By separating variables find the general solution to the differential equation in part (a).

Separating variables gives

$$\frac{dx}{(p-x)^2} = k\,dt$$

Integrating gives

$$\int \frac{dx}{(p-x)^2} = \int k\,dt = kt + C$$

Integrating the left hand side gives

$$\int \frac{dx}{(p-x)^2} = \frac{1}{p-x}$$

Thus,

$$\frac{1}{p-x} = kt + C \Rightarrow p - x = \frac{1}{kt+C} \Rightarrow x(t) = p - \frac{1}{kt+C}$$

[2] (c) Find the particular solution to the differential equation in part (a) subject to the initial condition $x(0) = 0$. Determine the limiting value of $x(t)$ as $t \to \infty$.

Applying the initial condition to the first form on the left above gives

$$\frac{1}{q} = C$$

Then the particular solution is

$$x(t) = p - \frac{1}{kt + \frac{1}{p}} = p - \frac{p}{pkt+1} = \frac{p^2kt + p - p}{pkt+1} = p\frac{pkt}{pkt+1}$$

The limiting value is given by

$$\lim_{t\to\infty} p\frac{pkt}{pkt+1} = \lim_{t\to\infty} p\frac{pk}{pk + \frac{1}{t}} = p$$

So the limiting value of the $x(t)$ is p. Of course, eventually every molecule of P is converted to the product.

[2] (d) The particular solution in part (c) above contains the product pk. Letting the time t be measured in hours, find the value of the product pk given that it takes one hour to reach 50% of the limiting value found on part (c) above. Then determine how long it will take to 90% of the limiting value found in part (c) above.

We are given that $x(1) = \frac{1}{2}p$. This gives

$$\frac{1}{2}p = pp\frac{pk}{pk+1} \Rightarrow \frac{1}{2} = p\frac{pk}{pk+1} = \frac{1}{2}pk + \frac{1}{2} = pk \Rightarrow pk = 1$$

So,

$$x(t) = p\frac{t}{t+1}$$

Now find the time when $x(t) = 0.9p$. This gives

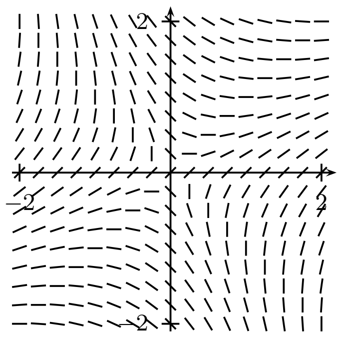
$$0.9p = p\frac{t}{t+1} \Rightarrow 0.9 = \frac{t}{t+1} \Rightarrow 0.9t + 0.9 = t \Rightarrow 0.1t = 0.9 \Rightarrow t = 9$$

So it takes 9 hours to reach 90% of the limiting value.

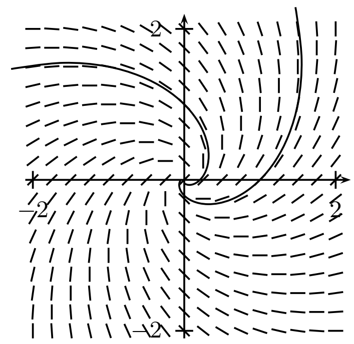
11 Consider the differential equation

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

[2] (a) Two slope fields are shown below. Pick the one that is the correct slope field for the differential equation above. Explain your choice.



(A)



(B)

Note that along the line $y = -x$ the slope given by the differential equation is zero. This is the case on (B) but not on (A).

[2] (b) On the slope field that you chose in part (a) draw the solutions for the given differential equation that satisfy the initial conditions: $y(0) = 1$ and $y(1) = 0$.

[3] (c) By completing the table below, use Euler's method to estimate $y(1.3)$ if $y(x)$ satisfies the differential equation above and $y(1) = 0$. Use a step size of 0.1.

x	y	$\frac{dy}{dx}$	$\frac{dy}{dx}\Delta x$
1.0	0.00000	1.0000	0.10000
1.1	0.10000	1.2000	0.12000
1.2	0.22000	1.4490	0.14490
1.3	0.36490		

[1] (d) Is the estimated solution in part (c) above an overestimate or an underestimate? Explain your answer.

The solution curve in part (b) through the point (1,0) is concave up, so the Euler's method estimate is an underestimate.

12 For each of the following infinite series

i) give the first four terms of the series

ii) determine whether the series converges or diverges

iii) if the series is geometric, give the sum of the series

[3] (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n} = \frac{1}{3} + \frac{4}{9} + \frac{9}{27} + \frac{16}{81} + \frac{25}{243} + \cdots$$

Here

$$a_n = \frac{n^2}{3^n} \qquad a_{n+1} = \frac{(n+1)^2}{3^{n+1}}$$

So that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^n}{n^2} \cdot \frac{(n+1)^2}{3^{n+1}}\right|$$
$$= \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3} < 1$$

Thus, the series converges, but it is not geometric.

[3] (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n! \cdot 2^n}{(2n+1)!}$

$$\sum_{n=1}^{\infty} \frac{n! \cdot 2^n}{(2n+1)!} = \frac{1}{3} + \frac{1}{15} + \frac{1}{105} + \frac{1}{945} + \cdots$$

Here

$$a_n = \frac{n! \cdot 2^n}{(2n+1)!} \qquad a_{n+1} = \frac{(n+1)! \cdot 2^{n+1}}{(2n+3)!}$$

So that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(2n+1)!}{n! \cdot 2^n} \cdot \frac{(n+1)! \cdot 2^{n+1}}{(2n+3)!}\right|$$
$$= \lim_{n\to\infty} \left(\frac{2(n+1)}{(2n+3)(2n+2)}\right) = \lim_{n\to\infty} \frac{1}{2n+3}$$
$$= 0 < 1$$

Thus, the series converges, but it is not geometric.

[3] (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{5^{n-1}}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{5^{n-1}} = 5 - 4 + \frac{16}{5} - \frac{64}{25} +$$

This is a geometric series with $r = -\frac{4}{5}$ and $a = 5$. Since $|r| < 1$ the series converges and the value is

$$\sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{5^{n-1}} = 5\left(\frac{1}{1 + \frac{4}{5}}\right) = \frac{25}{9}$$

13 Let $g(x) = \ln\left(\frac{x}{2-x}\right) = \ln x - \ln(2-x)$.

[3] (a) The Taylor series for a function f centered at $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(x-a)^n + \cdots$$

Find the first four non-zero terms in the Taylor series centered at $x = 1$ for the g defined above.

First compute the derivatives

$$\begin{aligned}
f(x) &= \ln x - \ln(2-x) & \Rightarrow \quad f(1) &= 0 \\
f'(x) &= \frac{1}{x} + \frac{1}{2-x} & \Rightarrow \quad f'(1) &= 2 \\
f''(x) &= -\frac{1}{x^2} + \frac{1}{(2-x)^2} & \Rightarrow \quad f''(1) &= 0 \\
f'''(x) &= \frac{2}{x^2} + \frac{2}{(2-x)^3} & \Rightarrow \quad f'''(1) &= 4 = 2 \cdot 2 \\
f^{(4)}(x) &= -\frac{2 \cdot 3}{x^4} + \frac{2 \cdot 3}{(2-x)^4} & \Rightarrow \quad f^{(4)}(1) &= 0 \\
f^{(5)}(x) &= \frac{2 \cdot 3 \cdot 4}{x^5} + \frac{2 \cdot 3 \cdot 4}{(2-x)^5} & \Rightarrow \quad f^{(5)}(1) &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 4
\end{aligned}$$

Thus,

$$g(x) = \ln\left(\frac{x}{2-x}\right) = 2(x-1) + \frac{1}{3!}(2 \cdot 2)(x-1)^3 + \frac{1}{5!}(2 \cdot 2 \cdot 2 \cdot 3 \cdot 4)(x-1)^5 + \cdots$$

[3] (b) Given that the series in part (a) above is

$$\sum_{n=0}^{\infty} \frac{2}{2n+1}(x-1)^{2n+1}$$

determine the radius of convergence of the series and give an open interval in which the series converges.

Here

$$a_n = \frac{2}{2n+1}(x-1)^{2n+1} \qquad\qquad a_{n+1} = \frac{2}{2n+3}(x-1)^{2n+3}$$

So that

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{2n+1}{2(x-1)^{2n+1}} \cdot \frac{2}{2n+3}(x-1)^{2n+3}\right| \\
&= \lim_{n\to\infty}\frac{2n+1}{2n+3}|x-2|^2 = |x-2|^2
\end{aligned}$$

The series converges if $|x-2|^2 < 1 \Rightarrow |x-2| < 1$. Thus, the radius of convergence is $R = 1$ and the open interval of convergence is $(0, 2)$.

[2] (c) Find the value of x for which

$$\frac{x}{2-x} = 2$$

and use your answer together with the series in part (a) above to estimate the value of $\ln 2$.

Solving the equation above gives

$$\frac{x}{2-x} = 2 \Rightarrow x = 4 - 2x \Rightarrow 3x = 4 \Rightarrow x = \frac{4}{3}$$

This value of x is in the interval of convergence found in part (b) above. Thus, noting that $\frac{4}{3} - 1 = \frac{1}{3}$

$$\begin{aligned}
\ln 2 = f\left(\frac{4}{3}\right) &\approx 2\left(\frac{1}{3}\right) + \frac{2}{3}\left(\frac{1}{3}\right)^3 + \frac{2}{5}\left(\frac{1}{3}\right)^5 + \cdots \\
&= \frac{2}{3} + \frac{2}{81} + \frac{2}{5 \cdot 3^5} + \frac{2}{7 \cdot 3^7} + \cdots \\
&= 0.69313 (\text{exact value is } \ln 2 = 0.69315) \\
&= \frac{2}{3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)3^{2n}}
\end{aligned}$$

A Short Table of Integrals

1. $\int f(g(x))g'(x)\,dx = \int f(u)\,du$ where $u = g(x)$
2. $\int u\,dv = vu - \int v\,du$
3. $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C$
4. $\int \frac{du}{u} = \ln|u| + C$
5. $\int e^u\,du = e^u + C$
6. $\int a^u\,du = \frac{1}{\ln a}a^u + C$
7. $\int \sin u\,du = -\cos u + C$
8. $\int \cos u\,du = \sin u + C$
9. $\int \sec^2 u\,du = \tan u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \tan u\,du = \ln|\sec u| + C$
12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + C$
14. $\int \frac{du}{a^2 + u^2} = \left(\frac{1}{a}\right)\arctan\left(\frac{u}{a}\right) + C$
15. $\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$
16. $\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$
17. $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$
18. $\int \sec^n u\,du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$
19. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2 + b^2}(a \sin bu - b \cos bu) + C$
20. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2 + b^2}(a \cos bu + b \sin bu) + C$
21. $\int \ln u\,du = u \ln u - u + C$

Some Identities

1. $\cos^2 x + \sin^2 x = 1$
2. $1 + \tan^2 x = \sec^2 x$
3. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
4. $\sin 2x = 2 \sin x \cos x$
5. $\cos^2 x = \frac{1 + \cos 2x}{2}$
6. $\sin^2 x = \frac{1 - \cos 2x}{2}$

Partial Fractions

1. $\frac{P(x)}{(ax+b)^k} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots \frac{A_k}{(ax+b)^k}$ where degree of P is less than k
2. $\frac{P(x)}{(px^2+qx+r)^k} = \frac{A_1 x + B_1}{px^2+qx+r} + \frac{A_2 x + B_2}{(px^2+qx+r)^2} + \cdots \frac{A_k x + B_k}{(px^2+qx+r)^k}$ where degree of P is less than $2k$

Numerical Integration

Midpoint Rule	$M_n = \Delta x\left[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)\right]$ where $\overline{x}_i = \frac{x_{i-1} + x_i}{2}$
Error in Midpoint Rule	$\lvert E_M\rvert \le \frac{K}{24}\frac{(b-a)^3}{n^2}$ where K is an upper bound on $\lvert f''(x)\rvert$ on $[a, b]$
Trapezoid Rule	$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$
Error in Trapezoid Rule	$\lvert E_T\rvert \le \frac{K}{12}\frac{(b-a)^3}{n^2}$ where K is an upper bound on $\lvert f''(x)\rvert$ on $[a, b]$
Simpson's Rule	$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$
Error in Simpson's Rule	$\lvert E_S\rvert \le \frac{K}{180}\frac{(b-a)^5}{n^4}$ where K is an upper bound on $\left\lvert f^{(4)}(x)\right\rvert$ on $[a, b]$