Student Name: ____________________

Student Number: ____________________

Total Marks: ________
100

Okanagan University College
Final Examination

Math 122 (Winter, 2003)

Instructor(s): Clint Lee

Section(s): 71 & 72

April 17, 2003

9:00AM

Duration: 3 hours

READ INSTRUCTIONS CAREFULLY BEFORE COMMENCING EXAM

INSTRUCTIONS: Answer all 13 questions in the spaces provided, showing all significant steps. Partial marks will be awarded for correct work even if the final answer is incorrect. Marks per question are given in the left margin, total 100. Check that your paper contains all 12 pages in addition to the cover page. The last page of the examination is a formula sheet. You may detach this sheet and use it in any of the problems on the exam.

This paper contains pages numbered 1 to 12

EXAM BOOKLETS ARE NOT REQUIRED

1 Evaluate each integral. Give the exact numerical value of any definite integral.

[3] (a) $\displaystyle\int \left(\frac{1}{x^2} + \frac{3}{3x+1} - e^{-x/2}\right) dx$

[3] (b) $\displaystyle\int_1^2 \frac{t^2}{\sqrt[3]{9-t^3}}\, dt$

[3] (c) $\displaystyle\int_0^{\pi/4} w^2 \cos 2w \, dw$

[3] (d) $\displaystyle\int \frac{x^2+x-1}{(x+2)(x-3)}\, dx$

2 During the first eight weeks after she is born a puppy grows at a rate $r(t)$, in kilograms per week, as shown in the graph at the right. The puppy weighed 0.5 kg when she was born.

[2] (a) Set up a definite integral giving $W(t)$, the puppy's weight t weeks after she was born.

[2] (b) Determine the puppy's weight

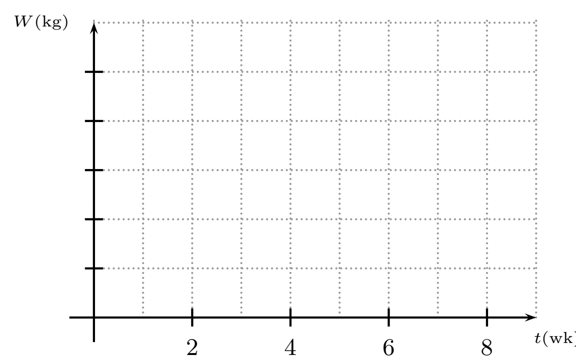
i) 2 weeks after she was born

ii) 4 weeks after she was born

iii) 6 weeks after she was born

iv) 8 weeks after she was born

[2] (c) Sketch a graph of $W(t)$ over the first 8 weeks after the puppy was born. Show the intervals where $W(t)$ is increasing and decreasing, the absolute maximum and minimum values, and the inflection point(s) of the graph.



[2] 3 (a) Write the definite integral $\int_{-1}^{2} \frac{e^x}{x+2}\,dx$ as a limit of the Riemann sum for the integral using right endpoints.

[2] (b) Write a definite integral given by the limit: $\lim_{n\to\infty} \sum_{i=1}^{n} \tan\left(\frac{\pi}{4} + \frac{\pi i}{12n}\right)\left(\frac{\pi}{12n}\right)$

4 The Q **integral** function is defined as

$$Q(x) = \int_0^x te^{-\cos t}\, dt$$

[2] (a) Find $\frac{d}{dx} Q(x)$.

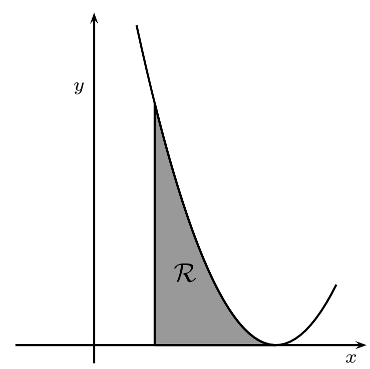
[3] (b) Verify that

$$\frac{d}{dx} Q(x)e^{\cos x} = -\sin x e^{\cos x}\, Q(x) + x$$

[2] (c) Verify that the function $y = Q(x)e^{\cos x} + Ce^{\cos x}$, where C is an arbitrary constant, is the general solution to the differential equation

$$\frac{dy}{dx} + (\sin x)\, y = x$$

5 The diagram shows the first quadrant region $\mathcal{R}$ bounded by $y = (x-3)^2$, $y = 0$, and $x = 1$.

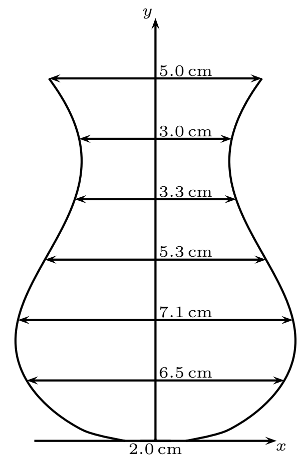


[3] (a) Use the disc method to find the volume of the solid generated when the region $\mathcal{R}$ is rotated about the x-axis.

[3] (b) Use the cylindrical shell method to find the volume of the solid generated when the region $\mathcal{R}$ is rotated about the y-axis.

[3] (c) Set up the integral to use the washer method to find the volume of the solid generated when the region $\mathcal{R}$ is rotated about the line $x = 4$. **Do not** simplify the integrand or evaluate the integral.

[2] (d) Set up the integral to use the cylindrical shell method to find the volume of the solid generated when the region $\mathcal{R}$ is rotated about the line $y = -2$. **Do not** simplify the integrand or evaluate the integral.



6 Jordan has bought a new vase and she wants to compute the volume of water that the vase holds. She measures the height of the vase to be 12 cm and diameter of the vase at 2 cm intervals. The diameter measurements are shown in the diagram. Note that the diameter at the base is 2 cm.

[2] (a) Let $d(y)$ be the diameter of the vase as a function of the distance y from its bottom. Set up an integral for the volume of the vase in terms of the function $d(y)$.

[4] (b) Use Simpson's rule to estimate value of the integral in part (a).

[3] 7 Use the Comparison Test to determine whether the integral below converges or diverges.

$$\int_0^\infty \frac{e^{-x}}{1+\sqrt{x}}\,dx$$

Hint: Explain why $1+\sqrt{x} \geq 1$ for all $x \geq 0$.

[3] 8 (a) Use an appropriate trigonometric substitution to evaluate the integral: $\displaystyle\int \frac{x^2}{(x^2+4)^{3/2}}\,dx$

[2] (b) Recall the hyperbolic functions $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$, and $\tanh x = \dfrac{\sinh x}{\cosh x}$. Further, recall that

$$\cosh^2 x - \sinh^2 x = 1, \quad \frac{d}{dx}\cosh x = \sinh x, \quad \frac{d}{dx}\sinh x = \cosh x$$

Use the definitions and identities above to show that

$$\frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x}$$

[4] (c) Make the hyperbolic substitution $x = 2\sinh t$ in the integral in part (a) above and evaluate the resulting integral using the results in part (b) above. Express the result in terms of x using the fact that

$$\sinh^{-1} z = \ln\left|z + \sqrt{z^2+1}\right|$$

[3] 9 (a) Use the substitution $z = x^2$ followed by integration by parts to evaluate the integral: $\int x^3 e^{-x^2}\, dx$

[2] (b) Use the result in part (a) above to evaluate the improper integral: $\int_0^\infty x^3 e^{-x^2}\, dx$

Recall that $\lim_{x\to\infty} x^n e^{-x} = 0$ for any n.

[3] (c) The *average speed* of molecules in an ideal gas is

$$\overline{v} = \frac{4}{\sqrt{\pi}} \left(\frac{M}{2RT}\right)^{3/2} \int_0^\infty v^3 e^{-Mv^2/(2RT)}\, dv$$

where M is the molecular weight of the gas, R is the gas constant, T is the gas temperature, and v is molecular speed. Make the substitution $x = \sqrt{\frac{M}{2RT}}v$ in this integral and use the result in part (b) above to show that

$$\overline{v} = \sqrt{\frac{8RT}{\pi M}}$$

10 A second order chemical reaction involves the the interaction of one molecule of a reactant P with one molecule a second reactant Q to produce one molecule of the product X. This is written $P+Q \to X$. Let p and q be the initial concentrations of the reactants P and Q, respectively, and $x(t)$ be the concentration of the product X at time t. Then $p - x(t)$ and $q - x(t)$ are the concentrations of reactants P and Q at time t. The rate at which the product X is produced is proportional to the product of the concentrations of the two reactants P and Q.

[2] (a) Assuming that $p = q$, write a differential equation for $x(t)$, the concentration of the product X at time t. Let k be the constant of proportionality.

[3] (b) By separating variables find the general solution to the differential equation in part (a).

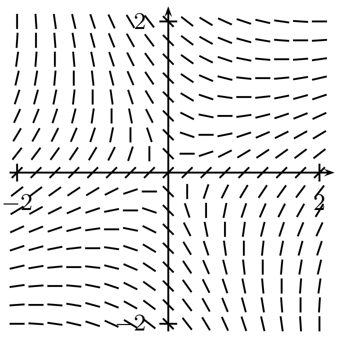
[2] (c) Find the particular solution to the differential equation in part (a) subject to the initial condition $x(0) = 0$. Determine the limiting value of $x(t)$ as $t \to \infty$.

[2] (d) The particular solution in part (c) above contains the product pk. Letting the time t be measured in hours, find the value of the product pk given that it takes one hour to reach 50% of the limiting value found on part (c) above. Then determine how long it will take to 90% of the limiting value found in part (c) above.

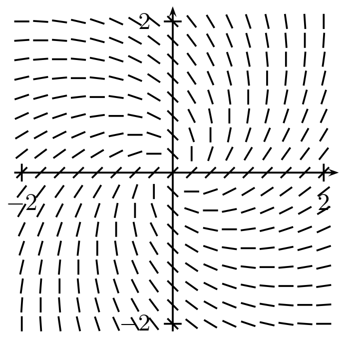
11 Consider the differential equation

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

[2] (a) Two slope fields are shown below. Pick the one that is the correct slope field for the differential equation above. Explain your choice.



(A)



(B)

[2] (b) On the slope field that you chose in part (a) draw the solutions for the given differential equation that satisfy the initial conditions: $y(0) = 1$ and $y(1) = 0$.

[3] (c) By completing the table below, use Euler's method to estimate $y(1.3)$ if $y(x)$ satisfies the differential equation above and $y(1) = 0$. Use a step size of 0.1.

x	y	$\frac{dy}{dx}$	$\frac{dy}{dx}\Delta x$
1.0	0.00000		
1.1			
1.2			
1.3			

[1] (d) Is the estimated solution in part (c) above an overestimate or an underestimate? Explain your answer.

12 For each of the following infinite series

i) give the first four terms of the series

ii) determine whether the series converges or diverges

iii) if the series is geometric, give the sum of the series

[3] (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

[3] (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n! \cdot 2^n}{(2n+1)!}$

[3] (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{5^{n-1}}$

13 Let $g(x) = \ln\left(\frac{x}{2-x}\right) = \ln x - \ln(2-x)$.

[3] (a) The Taylor series for a function f centered at $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(x-a)^n + \cdots$$

Find the first four non-zero terms in the Taylor series centered at $x = 1$ for the g defined above.

[3] (b) Given that the series in part (a) above is

$$\sum_{n=0}^{\infty} \frac{2}{2n+1}(x-1)^{2n+1}$$

determine the radius of convergence of the series and give an open interval in which the series converges.

[2] (c) Find the value of x for which

$$\frac{x}{2-x} = 2$$

and use your answer together with the series in part (a) above to estimate the value of $\ln 2$.

A Short Table of Integrals

1. $\int f(g(x))g'(x)\,dx = \int f(u)\,du$ where $u = g(x)$
2. $\int u\,dv = vu - \int v\,du$
3. $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C$
4. $\int \frac{du}{u} = \ln|u| + C$
5. $\int e^u\,du = e^u + C$
6. $\int a^u\,du = \frac{1}{\ln a}a^u + C$
7. $\int \sin u\,du = -\cos u + C$
8. $\int \cos u\,du = \sin u + C$
9. $\int \sec^2 u\,du = \tan u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \tan u\,du = \ln|\sec u| + C$
12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + C$
14. $\int \frac{du}{a^2 + u^2} = \left(\frac{1}{a}\right)\arctan\left(\frac{u}{a}\right) + C$
15. $\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$
16. $\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$
17. $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$
18. $\int \sec^n u\,du = \frac{1}{n-1}\tan u \sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$
19. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$
20. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$
21. $\int \ln u\,du = u\ln u - u + C$

Some Identities

1. $\cos^2 x + \sin^2 x = 1$
2. $1 + \tan^2 x = \sec^2 x$
3. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
4. $\sin 2x = 2\sin x \cos x$
5. $\cos^2 x = \frac{1 + \cos 2x}{2}$
6. $\sin^2 x = \frac{1 - \cos 2x}{2}$

Partial Fractions

1. $\frac{P(x)}{(ax+b)^k} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots \frac{A_k}{(ax+b)^k}$ where degree of P is less than k
2. $\frac{P(x)}{(px^2+qx+r)^k} = \frac{A_1x+B_1}{px^2+qx+r} + \frac{A_2x+B_2}{(px^2+qx+r)^2} + \cdots \frac{A_kx+B_k}{(px^2+qx+r)^k}$ where degree of P is less than $2k$

Numerical Integration

Midpoint Rule	$M_n = \Delta x\left[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)\right]$ where $\overline{x}_i = \frac{x_{i-1} + x_i}{2}$
Error in Midpoint Rule	$\lvert E_M\rvert \le \frac{K}{24}\frac{(b-a)^3}{n^2}$ where K is an upper bound on $\lvert f''(x)\rvert$ on $[a,b]$
Trapezoid Rule	$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$
Error in Trapezoid Rule	$\lvert E_T\rvert \le \frac{K}{12}\frac{(b-a)^3}{n^2}$ where K is an upper bound on $\lvert f''(x)\rvert$ on $[a,b]$
Simpson's Rule	$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$
Error in Simpson's Rule	$\lvert E_S\rvert \le \frac{K}{180}\frac{(b-a)^5}{n^4}$ where K is an upper bound on $\left\lvert f^{(4)}(x)\right\rvert$ on $[a,b]$